

# Bottleneck Matchings and Hamiltonian Cycles in Higher-Order Gabriel Graphs\*

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## Abstract

Given a set  $P$  of  $n$  points in the plane, the order- $k$  Gabriel graph on  $P$ , denoted by  $k$ - $GG$ , has an edge between two points  $p$  and  $q$  if and only if the closed disk with diameter  $pq$  contains at most  $k$  points of  $P$ , excluding  $p$  and  $q$ . It is known that 10- $GG$  contains a Euclidean bottleneck matching of  $P$ , while 8- $GG$  may not contain such a matching. We answer the following question in the affirmative: does 9- $GG$  contain any Euclidean bottleneck matching of  $P$ ? Thereby, we close the gap for the containment problem of Euclidean bottleneck matchings in Gabriel graphs.

It is also known that 10- $GG$  contains a Euclidean bottleneck Hamiltonian cycle of  $P$ , while 5- $GG$  may not contain such a cycle. We improve the lower bound and show that 7- $GG$  may not contain any Euclidean bottleneck Hamiltonian cycle of  $P$ .

## 1 Introduction

Let  $P$  be a set of  $n$  points in the plane. For any two points  $p, q \in P$ , let  $D[p, q]$  denote the closed disk that has the line segment  $\overline{pq}$  as diameter. Let  $|pq|$  be the Euclidean distance between  $p$  and  $q$ . The *Gabriel graph* on  $P$ , denoted by  $GG(P)$ , is a geometric graph that has an edge between two points  $p$  and  $q$  if and only if  $D[p, q]$  does not contain any point of  $P \setminus \{p, q\}$ . Gabriel graphs were introduced by Gabriel and Sokal [7] and can be computed in  $O(n \log n)$  time [10]. Every Gabriel graph has at most  $3n - 8$  edges, for  $n \geq 5$ , and this bound is tight [10].

Gabriel graphs are a family of proximity graphs that have been well studied in computational geometry. Beside their interesting graph theoretical properties, they have applications in several fields such as geographic information systems, computer graphics, or mesh generation (see, for example, [8, 10, 11]).

The *order- $k$  Gabriel graph* on  $P$ , denoted by  $k$ - $GG$ , is the geometric graph that has an edge between two points  $p$  and  $q$  if and only if  $D[p, q]$  contains at most  $k$  points of  $P \setminus \{p, q\}$ . Thus, the Gabriel graph,  $GG(P)$ , corresponds to 0- $GG$ . Su and Chang [11] showed that  $k$ - $GG$  can be constructed in  $O(k^2 n \log n)$  time and contains  $O(k(n - k))$  edges. For two points  $p, q \in P$ , the *lune* of  $p$  and  $q$ , denoted by  $L(p, q)$ , is defined as the intersection of the two open disks of radius  $|pq|$  centered at  $p$  and  $q$ . The *order- $k$  Relative Neighborhood Graph* on  $P$ , denoted by  $k$ - $RNG$ , is the geometric graph that has an edge  $(p, q)$  if and only if  $L(p, q)$  contains at most  $k$  points of  $P$ . Note that  $k$ - $RNG$  on  $P$  is a subgraph of  $k$ - $GG$  on  $P$ .

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A *matching* in a graph  $G$  is a set of edges without common vertices. A *perfect matching* is a matching that matches all the vertices of  $G$ . A *Hamiltonian cycle* in  $G$  is a cycle that visits each vertex of  $G$  exactly once. In the case when  $G$  is an edge-weighted graph, a *bottleneck matching* is defined to be a perfect matching in  $G$ , in which the weight of the maximum-weight edge is minimized. Moreover, a *bottleneck Hamiltonian cycle* is a Hamiltonian cycle in  $G$ , in which the weight of the maximum-weight edge is minimized. For a point set  $P$ , a *Euclidean bottleneck matching* is a perfect matching in the complete graph with vertex set  $P$  that minimizes the longest edge; the weight of an edge is defined to be the Euclidean distance between its two endpoints. Similarly, a *Euclidean bottleneck Hamiltonian cycle* is a Hamiltonian cycle that minimizes the longest edge.

Beside other interesting theoretical properties of higher-order Gabriel graphs, their Hamiltonicity and “having a perfect matching” have been well studied, e.g., see [1, 2, 3, 4, 5, 9]. Chang et al. [5] proved that a Euclidean bottleneck matching of  $P$  is contained in 16-*RNG*.<sup>1</sup> This implies that 16-*GG* contains a Euclidean bottleneck matching. In [2] the authors improved the bound for the latter graphs by showing that 10-*GG* contains a Euclidean bottleneck matching. They also show that 8-*GG* may not have any Euclidean bottleneck matching. They asked if 9-*GG* contains any Euclidean bottleneck matching. In Section 2, we answer this question in the affirmative, and thus close the gap for the containment problem of Euclidean bottleneck matchings in Gabriel graphs..

**Theorem 1.** *For every point set  $P$ , 9- $GG$  contains a Euclidean bottleneck matching of  $P$ .*

Chang et al. [4] proved that a Euclidean bottleneck Hamiltonian cycle of  $P$  is contained in 19-*RNG*, which implies that 19-*GG* contains a Euclidean bottleneck Hamiltonian cycle. Abellanas et al. [1] improved the bound by showing that 15-*GG* contains a Euclidean bottleneck Hamiltonian cycle. Kaiser et al. [9] improved the bound further by showing that 10-*GG* contains a Euclidean bottleneck Hamiltonian cycle. They also provide an example which shows that 5-*GG* may not contain any Euclidean bottleneck Hamiltonian cycle. In Section 3, we improve the lower bound to 7 and prove Proposition 1. For the Hamiltonicity of Gabriel graphs that are defined by convex shapes, other than disks, we refer an interested reader to the result of Bose et al. [3].

**Proposition 1.** *There exist point sets  $P$  such that 7- $GG$  does not contain any Euclidean bottleneck Hamiltonian cycle of  $P$ .*

Therefore, it remains open to decide whether or not 8-*GG* or 9-*GG* contains a Euclidean bottleneck Hamiltonian cycle.

## 2 Proof of Theorem 1

In this section we prove Theorem 1. We combine techniques from [5], [6] and [9] in our proof and close the gap for the containment problem of Euclidean bottleneck matchings in Gabriel graphs. A similar combination has been used by Kaiser et al. [9] for Euclidean bottleneck Hamiltonian cycles. The proofs of Lemmas 1 and 2 are similar to that of [5] for relative neighborhood graphs. The proof of Lemma 3 is based on a similar technique that is used in [9] for the Hamiltonicity of 10-*GG*.

Let  $\mathcal{M}$  be the set of all perfect matchings of the complete graph with vertex set  $P$ . For a matching  $M \in \mathcal{M}$  we define the *weight sequence* of  $M$ ,  $WS(M)$ , as the sequence containing the weights of the edges of  $M$  in non-increasing order. A matching  $M_1$  is said to be less than

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<sup>1</sup>They defined *k-RNG* to have an edge  $(p, q)$  if and only if  $L(p, q)$  contains at most  $k - 1$  points of  $P$ .

a matching  $M_2$  if  $\text{WS}(M_1)$  is lexicographically smaller than  $\text{WS}(M_2)$ . We define a total order on the elements of  $\mathcal{M}$  by their weight sequence. If two elements have exactly the same weight sequence, break ties arbitrarily to get a total order.

Let  $M^* = \{(a_1, b_1), \dots, (a_{\frac{n}{2}}, b_{\frac{n}{2}})\}$  be a matching in  $\mathcal{M}$  with minimum weight sequence. Observe that  $M^*$  is a Euclidean bottleneck matching for  $P$ . In order to prove Theorem 1, we will show that all edges of  $M^*$  are in 9- $GG$ . Consider any edge  $(a, b)$  in  $M^*$ . If  $D[a, b]$  contains no point of  $P \setminus \{a, b\}$ , then  $(a, b)$  is an edge of 9- $GG$ . Suppose that  $D[a, b]$  contains  $k$  points of  $P \setminus \{a, b\}$ . We are going to prove that  $k \leq 9$ . Let  $R = \{r_1, r_2, \dots, r_k\}$  be the set of points of  $P \setminus \{a, b\}$  that are in  $D[a, b]$ . Let  $S = \{s_1, s_2, \dots, s_k\}$  represent the points for which  $(r_i, s_i) \in M^*$ .

Without loss of generality, we assume that  $D[a, b]$  has diameter 1 and is centered at the origin  $o = (0, 0)$ , and  $a = (-0.5, 0)$  and  $b = (0.5, 0)$ . For any point  $p$  in the plane, let  $\|p\|$  denote the distance of  $p$  from  $o$ . Note that  $|ab| = 1$ , and for any point  $x \in D[a, b] \setminus \{a, b\}$  we have  $\max\{|xa|, |xb|\} < 1$ .

**Lemma 1.** *For each point  $s_i \in S$ ,  $\min\{|s_i a|, |s_i b|\} \geq 1$ .*

*Proof.* The proof is by contradiction; suppose that  $|s_i a| < 1$ . Let  $M$  be the perfect matching obtained from  $M^*$  by deleting  $\{(a, b), (r_i, s_i)\}$  and adding  $\{(s_i, a), (r_i, b)\}$ . The lengths of the two new edges are smaller than 1, and hence both  $(s_i, a)$  and  $(r_i, b)$  are shorter than  $(a, b)$ . Thus,  $\text{WS}(M) <_{\text{lex}} \text{WS}(M^*)$ , which contradicts the minimality of  $M^*$ .  $\square$

As a corollary of Lemma 1,  $R$  and  $S$  are disjoint.

**Lemma 2.** *For each pair of points  $s_i, s_j \in S$ ,  $|s_i s_j| \geq \max\{|r_i s_i|, |r_j s_j|, 1\}$ .*

*Proof.* The proof is by contradiction; suppose that  $|s_i s_j| < \max\{|r_i s_i|, |r_j s_j|, 1\}$ . Let  $M$  be the perfect matching obtained from  $M^*$  by deleting  $\{(a, b), (r_i, s_i), (r_j, s_j)\}$  and adding  $\{(a, r_i), (b, r_j), (s_i, s_j)\}$ . Note that  $\max\{|ar_i|, |br_j|, |s_i s_j|\} < \max\{|r_i s_i|, |r_j s_j|, |ab|\}$ . Thus,  $\text{WS}(M) <_{\text{lex}} \text{WS}(M^*)$ , which contradicts the minimality of  $M^*$ .  $\square$

Let  $C(x, r)$  (resp.  $D(x, r)$ ) be the circle (resp. closed disk) of radius  $r$  that is centered at a point  $x$  in the plane. For  $i \in \{1, \dots, k\}$ , let  $s'_i$  be the intersection point between  $C(o, 1.5)$  and the ray with origin at  $o$  passing through  $s_i$ . Let the point  $p_i$  be  $s_i$ , if  $\|s_i\| < 1.5$ , and  $s'_i$ , otherwise. See Figure 1. Let  $T = \{a, b, p_1, \dots, p_k\}$ .

**Observation 1.** *Let  $s_j$  be a point in  $S$ , where  $\|s_j\| \geq 1.5$ . Then, the disk  $D(s_j, \|s_j\| - 0.5)$  is contained in the disk  $D(s_j, |s_j r_j|)$ . Moreover, the disk  $D(p_j, 1)$  is contained in the disk  $D(s_j, \|s_j\| - 0.5)$ . See Figure 1.*

**Lemma 3.** *The distance between any pair of points in  $T$  is at least 1.*

*Proof.* Let  $x$  and  $y$  be two points in  $T$ . We are going to prove that  $|xy| \geq 1$ . We distinguish between the following three cases.

- $\{x, y\} = \{a, b\}$ . In this case the claim is trivial.
- $x \in \{a, b\}, y \in \{p_1, \dots, p_k\}$ . If  $\|y\| = 1.5$ , then  $y$  is on  $C(o, 1.5)$ , and hence  $|xy| \geq 1$ . If  $\|y\| < 1.5$ , then  $y$  is a point in  $S$ . Therefore, by Lemma 1,  $|xy| \geq 1$ .
- $x, y \in \{p_1, \dots, p_k\}$ . Without loss of generality assume  $x = p_i$  and  $y = p_j$ , where  $1 \leq i < j \leq k$ . We differentiate between three cases:

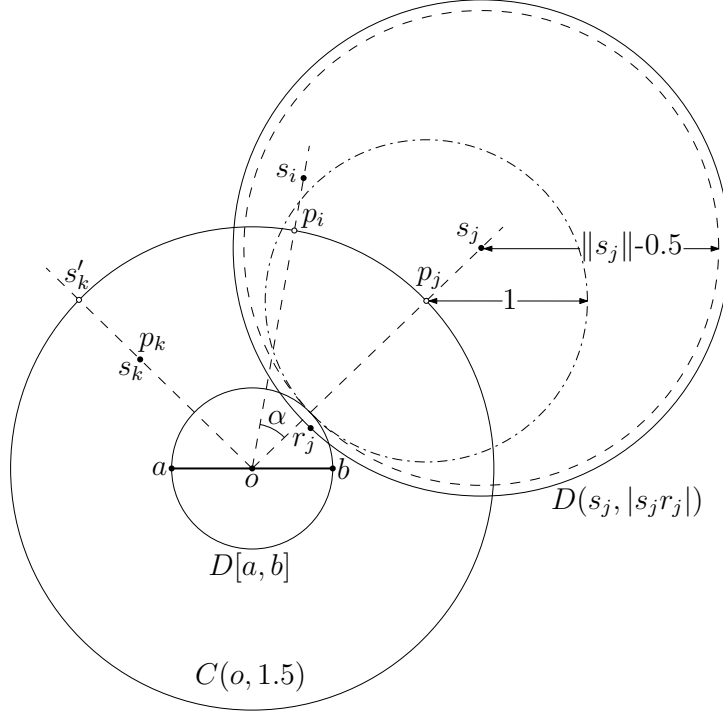


Figure 1: Proof of Lemma 3;  $p_i = s'_i$ ,  $p_j = s'_j$ , and  $p_k = s_k$ .

*Case (i):*  $\|p_i\| < 1.5$  and  $\|p_j\| < 1.5$ . In this case  $p_i$  and  $p_j$  are two points in  $S$ . Therefore, by Lemma 2,  $|p_i p_j| \geq 1$ .

*Case (ii):*  $\|p_i\| < 1.5$  and  $\|p_j\| = 1.5$ . In this case  $p_i$  is a point in  $S$ . By Observation 1, the disk  $D(p_j, 1)$  is contained in the disk  $D(s_j, |s_j r_j|)$ , and by Lemma 2,  $p_i$  is not in the interior of  $D(s_j, |s_j r_j|)$ . Therefore,  $p_i$  is not in the interior of  $D(p_j, 1)$ , which implies that  $|p_i p_j| \geq 1$ .

*Case (iii):*  $\|p_i\| = 1.5$  and  $\|p_j\| = 1.5$ . In this case  $\|s_i\| \geq 1.5$  and  $\|s_j\| \geq 1.5$ . Without loss of generality assume  $\|s_i\| \leq \|s_j\|$ . For the sake of contradiction assume that  $|p_i p_j| < 1$ ; see Figure 1. Then, for the angle  $\alpha = \angle s_i o s_j$  we have  $\sin(\alpha/2) < \frac{1}{3}$ . Then,  $\cos(\alpha) = 1 - 2\sin^2(\alpha/2) > \frac{7}{9}$ . By the law of cosines in the triangle  $\triangle s_i o s_j$ , we have

$$|s_i s_j|^2 < \|s_i\|^2 + \|s_j\|^2 - \frac{14}{9} \|s_i\| \|s_j\|. \quad (1)$$

By Observation 1, the disk  $D(s_j, \|s_j\| - 0.5)$  is contained in the disk  $D(s_j, |s_j r_j|)$ , and by Lemma 2,  $s_i$  is not in the interior of  $D(s_j, |s_j r_j|)$ . Therefore,  $s_i$  is not in the interior of  $D(s_j, \|s_j\| - 0.5)$ . Thus,  $|s_i s_j| \geq \|s_j\| - 0.5$ . In combination with Inequality (1), this implies

$$\|s_j\| \left( \frac{14}{9} \|s_i\| - 1 \right) < \|s_i\|^2 - \frac{1}{4}. \quad (2)$$

In combination with the assumption  $\|s_i\| \leq \|s_j\|$ , Inequality (2) implies

$$\frac{5}{9} \|s_i\|^2 - \|s_i\| + \frac{1}{4} < 0,$$

i.e.,

$$\frac{5}{9} \left( \|s_i\| - \frac{3}{10} \right) \left( \|s_i\| - \frac{3}{2} \right) < 0.$$

This is a contradiction, because, since  $\|s_i\| \geq 1.5$ , the left-hand side is non-negative. Thus  $|p_i p_j| \geq 1$ , which completes the proof of the lemma. □

By Lemma 3, the points in  $T$  have mutual distance at least 1. Moreover, the points in  $T$  lie in  $D(o, 1.5)$ . Fodor [6] proved that the smallest circle which contains 12 points with mutual distances at least 1 has radius 1.5148. Therefore,  $T$  contains at most 11 points. Since  $a, b \in T$ , this implies that  $k \leq 9$ . Therefore,  $S$ , and consequently  $R$ , contains at most 9 points. Thus,  $(a, b)$  is an edge in 9- $GG$ . This completes the proof of Theorem 1.

### 3 Proof of Proposition 1

In this section we prove Proposition 1. We show that for some point sets  $P$ , 7- $GG$  does not contain any Euclidean bottleneck Hamiltonian cycle of  $P$ .

Figure 2 shows a configuration of a set  $P = \{a, b, r_1, \dots, r_8, s_1, \dots, s_{15}\}$  of 25 points. The closed disk  $D[a, b]$  is centered at  $o$  and has diameter one, i.e.,  $|ab| = 1$ .  $D[a, b]$  contains all 8 points of the set  $R = \{r_1, \dots, r_8\}$ ; these points lie on the circle with radius  $\frac{1}{2} - \epsilon$  that is centered at  $o$ ; all points of  $R$  are in the interior of  $D[a, b]$ . The 15 red circles have radius 1 and centered at points of the set  $S = \{s_1, \dots, s_{15}\}$ . The points of  $S$  are placed in such a way that the distance between each of them to its first and second closest point is exactly 1. These distances are illustrated by bold black edges in Figure 2; let  $B$  be the chain formed by these edges. Note that  $r_1$  and  $r_8$  are the endpoints of  $B$ , and  $|r_1 s_1| = |r_8 s_7| = 1$ . For each point  $s_i$ , we have  $|s_i a| > 1$  and  $|s_i b| > 1$ . Consider the Hamiltonian cycle  $H = B \cup \{(r_1, a), (a, b), (b, r_8)\}$ . The longest edge in  $H$  has length 1. Therefore, the length of the longest edge in any bottleneck Hamiltonian cycle for  $P$  is at most 1. We are going to show that any bottleneck Hamiltonian cycle of  $P$  contains  $(a, b)$ . Since in  $B$  each point of  $S$  is connected to its first and second closest point, every bottleneck Hamiltonian cycle of  $P$  contains  $B$ , because otherwise, one of the points in  $S$  should be connected to a point that is farther than its second closest point, and hence that edge is longer than 1. The only way to finish a bottleneck Hamiltonian cycle, using the endpoints of  $B$  and the points  $a$  and  $b$ , is to have the edge  $(a, b)$ . Therefore, any bottleneck Hamiltonian cycle of  $P$ , contains  $(a, b)$ . Since  $D[a, b]$  contains 8 points of  $P \setminus \{a, b\}$ , we get that  $(a, b) \notin 7\text{-}GG$ . Therefore 7- $GG$  does not contain any Euclidean bottleneck Hamiltonian cycle of  $P$ .

### 4 Conclusion

We considered the inclusion of a Euclidean bottleneck matching and a Euclidean bottleneck Hamiltonian cycle of a point set  $P$  in higher order Gabriel graphs. It is known that 10- $GG$  contains a bottleneck matching and a bottleneck Hamiltonian cycle of  $P$ . We proved that 9- $GG$  contains a bottleneck matching of  $P$  and 7- $GG$  may not contain any bottleneck Hamiltonian cycle of  $P$ . It remains open to decide if 8- $GG$  or 9- $GG$  contains any bottleneck Hamiltonian cycle of  $P$ .

### References

- [1] M. Abellanas, P. Bose, J. García-López, F. Hurtado, C. M. Nicolás, and P. Ramos. On structural and graph theoretic properties of higher order Delaunay graphs. *Int. J. Comput. Geometry Appl.*, 19(6):595–615, 2009.

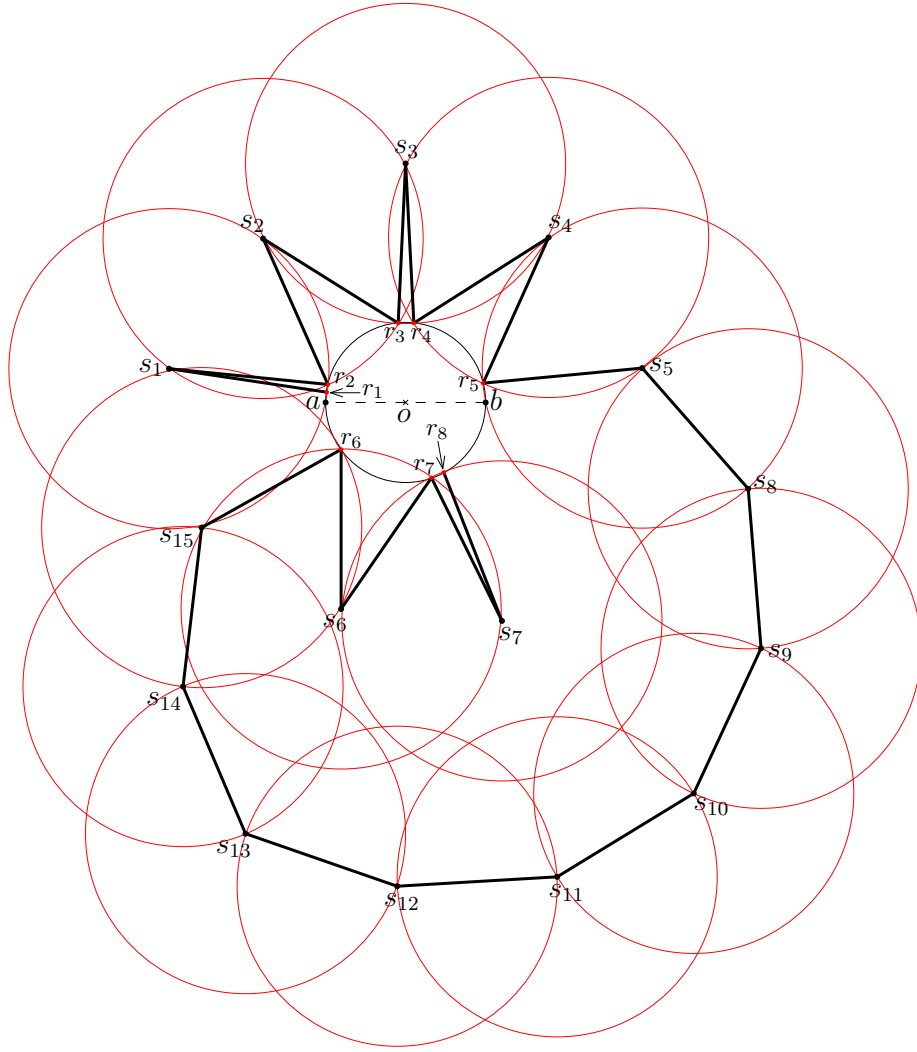


Figure 2: Proof of Proposition 1. The bold-black edges belong to  $B$ .  $D[a, b]$  contains 8 points.

- [2] A. Biniarz, A. Maheshwari, and M. Smid. Matchings in higher-order Gabriel graphs. *Theor. Comput. Sci.*, 596:67–78, 2015.
- [3] P. Bose, P. Cano, M. Saumell, and R. I. Silveira. Hamiltonicity for convex shape Delaunay and Gabriel graphs. In *the 35th European Workshop on Computational Geometry (EuroCG)*, 2019.
- [4] M.-S. Chang, C. Y. Tang, and R. C. T. Lee. 20-relative neighborhood graphs are Hamiltonian. *Journal of Graph Theory*, 15(5):543–557, 1991.
- [5] M.-S. Chang, C. Y. Tang, and R. C. T. Lee. Solving the Euclidean bottleneck matching problem by  $k$ -relative neighborhood graphs. *Algorithmica*, 8(3):177–194, 1992.
- [6] F. Fodor. The densest packing of 12 congruent circles in a circle. *Beiträge Algebra Geom*, 41:401–409, 2000.
- [7] K. R. Gabriel and R. R. Sokal. A new statistical approach to geographic variation analysis. *Systematic Zoology*, 18(3):259–278, 1969.
- [8] J. W. Jaromczyk and G. T. Toussaint. Relative neighborhood graphs and their relatives. *Proceedings of the IEEE*, 80(9):1502–1517, 1992.
- [9] T. Kaiser, M. Saumell, and N. V. Cleemput. 10-Gabriel graphs are Hamiltonian. *Inf. Process. Lett.*, 115(11):877–881, 2015.

- [10] D. W. Matula and R. R. Sokal. Properties of Gabriel graphs relevant to geographic variation research and the clustering of points in the plane. *Geographical Analysis*, 12(3):205–222, 1980.
- [11] T.-H. Su and R.-C. Chang. The  $k$ -Gabriel graphs and their applications. In *SIGAL International Symposium on Algorithms*, pages 66–75, 1990.