Communication-Efficient Construction of the Plane Localized Delaunay Graph

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Abstract

Let V be a finite set of points in the plane. We present a 2-local algorithm that constructs a plane $\frac{4\pi\sqrt{3}}{9}$-spanner of the unit-disk graph $UDG(V)$. Each node can only communicate with nodes that are within unit-distance from it. This algorithm only makes one round of communication and each point of V broadcasts at most 5 messages. This improves on all previously known message-bounds for this problem.

1 Introduction

A *wireless ad hoc network* consists of a finite set V of wireless nodes. Each node u in V is a point in the plane that can communicate directly with all points of V within u's communication range. If this range is one unit for each point, then the network is modeled by the *unit-disk graph* $UDG(V)$ of V. This (undirected) graph has V as its vertex set and any two distinct vertices u and v are connected by an edge if and only if the Euclidean distance $|uv|$ between u and v is at most one unit.

In order for two points that are more than one unit apart to be able to communicate, the points of V use a so-called *local algorithm* (to be defined below) to construct a subgraph G of $UDG(V)$. This subgraph should have the property that it supports efficient routing of messages, i.e., there should be a simple and efficient protocol that allows any point of V to send a message to any other point of V.

In this paper, we present a local algorithm that constructs a subgraph G of $UDG(V)$ that satisfies the following properties:

1. Each point u of V stores a set $E(u)$ of edges that are incident on u. The edge set of G is equal to $\cup_{u \in V} E(u)$.

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2. The edge sets $E(u)$ with $u \in V$ are *consistent*: For any two points u and v in V, (u, v) is an edge in $E(u)$ if and only if (u, v) is an edge in $E(v)$.

3. The graph G is *plane*: If we consider each edge (u, v) to be the straight-line segment joining u and v, then no two edges of G cross[1]. The graph being plane is useful, because several algorithms are known for routing messages in a plane subgraph of $UDG(V)$; see, e.g., Bose *et al.* [3] and Karp and Kung [6].

4. The graph G is a *t-spanner* of $UDG(V)$, for some constant $t > 1$: For each edge (u, v) of $UDG(V)$, the graph G contains a path between u and v whose Euclidean length is at most $t|uv|$. Observe that this implies that shortest-path distances in $UDG(V)$ are approximated, within a factor of t, by shortest-path distances in G. Thus, this property implies that the total distance traveled by a message, when using G, is not much larger than the minimum distance that needs to be traveled in $UDG(V)$. Our construction shows that $t \leq \frac{4\pi\sqrt{3}}{9}$.

1.1 Local Algorithms

As mentioned above, we model a wireless ad hoc network by the unit-disk graph $UDG(V)$, where V is a finite set of points in the plane. The points of V want to construct a communication graph G (which is a subgraph of $UDG(V)$) using a distributed and local algorithm. In this section, we formalize this notion and introduce the complexity measures that we will use to analyze the efficiency of such algorithms.

The points of V can communicate with each other by broadcasting messages. If a point u of V broadcasts a message, then each point of V within Euclidean distance one from u receives the message. Each point of V can perform computations based on its location and all information received from other points. Informally, an algorithm is called *local*, if the computation performed at each point u of V is based only on its location and the locations of all points that are within distance k (in $UDG(V)$) from u, for some small integer $k \geq 1$. Thus, in a local algorithm, information cannot "travel" over a "large" distance.

To define this notion formally, let $\delta_{UDG}(u, v)$ denote the Euclidean length of a shortest path between the points u and v in the graph $UDG(V)$. For any integer $k \geq 1$, let

$$N_k(u) = \{v \in V : \delta_{UDG}(u, v) \leq k\}.$$

Observe that $u \in N_k(u)$.

Let $\mathcal{A}(V)$ be a distributed algorithm that runs on a set V of points in the plane, and let $\mathcal{A}(u; V)$ denote the computation performed by point u. As is common in this field, we assume that, at the start of the algorithm, each point u of V knows the locations (i.e., the x- and y-coordinates) of all points in $N_1(u)$. Thus, the set $N_1(u)$ can be considered to be the input for u.

[1] Two edges are said to *cross* if they are not collinear and there exists a (unique) point that is in the relative interior of both edges.

For any point u of V, we denote by $T_u(V)$ the *trace* of the computation performed by $\mathcal{A}(u;V)$. Thus, $T_u(V)$ contains the sequence of all computing and broadcasting operations performed by $\mathcal{A}(u;V)$ when each point v of V runs algorithm $\mathcal{A}(v;V)$.

Definition 1 *For an integer $k \geq 1$, we say that $\mathcal{A}(V)$ is a k-*local *algorithm, if for each point u of V,*

$$T_u(V) = T_u(N_k(u)).$$

In other words, for every point u of V, the following holds: If we run the *entire* distributed algorithm $\mathcal{A}$ with V replaced by $N_k(u)$, then the computation performed by u does not change (even though the computations performed by other points may change).

A k-local algorithm runs in parallel on all points of V, where each point u performs an alternating sequence of computation steps and broadcasting steps in a synchronized manner.

1. In a *computation step*, point u performs some computation based on the subset of $N_k(u)$ that is known to u at that moment. For example, u may compute the Delaunay triangulation of this subset; we consider this to be one computation step. We assume that each such computation step works in the algebraic computation model.

2. In a *broadcasting step*, point u broadcasts a (possibly empty) sequence of messages, which is received by all points in $N_1(u)$.

3. A *message* is defined to be the location of a point in the plane. A message broadcast by u need not be an element of V, but it must have been computed, based on the subset of $N_k(u)$ that is known to u at that moment, in the algebraic computation model. (Thus, bit-manipulation cannot be used to encode several points, or any other information, in one message.)

The efficiency of a local algorithm will be expressed in terms of the following measures:

1. The value of k. The smaller the value of k, the “more local” the algorithm is.

2. The maximum number of messages that are broadcast by any point of V. The goal is to minimize this number.

3. The number of *communication rounds*, which is defined to be the maximum number of broadcasting steps performed by any point in V. This number measures the (parallel) time for the entire algorithm to complete its computation. Again, the goal is to minimize this number.

1.2 Previous Work

Above, we have defined the notion of a t-spanner of the unit-disk graph $UDG(V)$. For a real number $t > 1$, a graph G is called a t-spanner of the *point set* V if for any two elements u and v of V, there exists a path in G between u and v whose length is at most

$t|uv|$. The problem of constructing t-spanners for point sets has been studied intensively in computational geometry; see the book by Narasimhan and Smid [9] for a survey.

Since we are concerned with plane spanners of the unit-disk graph, our algorithm will be based on the *Delaunay Triangulation* $DT(V)$ of V; see, e.g., the textbook by de Berg *et al.* [4]. Recall that $DT(V)$ is the plane graph with vertex set V in which any two distinct points u and v are connected by an edge if and only if there exists a disk D such that (i) u and v are the only points of V that are on the boundary of D and (ii) no point of V is in the interior of D. Also, three points u, v, and w determine a triangular face of $DT(V)$ if and only if the disk having u, v, and w on its boundary does not contain any point of V in its interior. Keil and Gutwin [7] have shown that $DT(V)$ is a $\frac{4\pi\sqrt{3}}{9}$-spanner of V. To extend this result to unit-disk graphs, it is natural to consider subgraphs of $UDel(V)$, which is defined to be the intersection of the Delaunay triangulation and the unit-disk graph of V. It has been shown by Bose *et al.* [2] that $UDel(V)$ is a $\frac{4\pi\sqrt{3}}{9}$-spanner of $UDG(V)$. Unfortunately, constructing $UDel(V)$ using a k-local algorithm, for any constant value of k, is not possible: Consider an edge (u, v) in $UDel(V)$ whose empty disk D is very large. In order for a k-local algorithm to verify that no point of V is in the interior of D, information about the points of V must travel over a large distance to u or v. Clearly, this is possible only if the value of k is very large. Because of this, researchers have considered the problem of designing local algorithms that construct a plane subgraph of $UDG(V)$ which is a *supergraph* of $UDel(V)$. Obviously, by the result of [2], such a graph is also a $\frac{4\pi\sqrt{3}}{9}$-spanner of $UDG(V)$.

Gao *et al.* [5] proposed a 2-local algorithm that constructs a plane subgraph of $UDG(V)$ which is a supergraph of $UDel(V)$. However, the number of messages broadcast by a single point of V can be as large as $\Theta(n)$, where n is the number of elements of V. This result was improved by Li *et al.* [8]: They presented a 2-local algorithm that constructs such a graph in four communication rounds and in which each point broadcasts at most 49 messages.

Currently, the best result for computing a plane t-spanner (for some constant t) of the unit-disk graph $UDG(V)$ is by Araujo and Rodrigues [1]. They presented a 2-local algorithm which computes such a spanner in one communication round and in which each point broadcasts at most 11 messages.

1.3 Our Result

In this paper, we modify the algorithm of Araujo and Rodrigues [1] and improve the upper bound on the message complexity for each point of V from 11 to 5:

Theorem 1 *Let V be a finite set of points in the plane. There exists a 2-local algorithm that computes a plane and consistent $\frac{4\pi\sqrt{3}}{9}$-spanner of the unit-disk graph of V. This algorithm makes one communication round and each point of V broadcasts at most 5 messages.*

The rest of this paper is organized as follows. In Section 2, we present a preliminary 2-local algorithm that computes, in one communication round, a subgraph of $UDG(V)$. In this algorithm, each point of V broadcasts at most 6 messages. We present a *rigorous* proof of the fact that the graph computed by this algorithm is a plane and consistent $\frac{4\pi\sqrt{3}}{9}$-spanner of $UDG(V)$. In Section 3, we make a simple modification to the algorithm of Section 2 which reduces the message complexity for each point of V from 6 to 5. We then show that the new algorithm and the algorithm of Section 2 compute the same graph. Thus, this will prove Theorem 1. We conclude in Section 4 with some directions for future work.

Throughout the rest of this paper, we assume that the points in the set V are in general position (meaning that no three points of V are collinear and no four points of V are co-circular). We also assume that the unit-disk graph $UDG(V)$ is connected. We will use the following notation:

- $D(a, b, c)$ denotes the disk having the three points a, b, and c on its boundary.
- $D(c; r)$ denotes the disk centered at the point c and having radius r.
- $\Delta(a, b, c)$ denotes the triangle having the three points a, b, and c as its vertices.
- ∂D denotes the boundary of the disk D.
- $int(D)$ denotes the interior of the disk D.
- Let v, x, and y be points of V, where $v \neq y$. Assume there exists a disk D such that $N_1(x) \cap \partial D = \{v, y\}$ and $N_1(x) \cap int(D) = \emptyset$. We denote such a disk D by $Del_x(v, y)$. Observe that $Del_x(v, y)$ is a certificate for the fact that (v, y) is an edge in the Delaunay triangulation of the point set $N_1(x)$.

2 A Preliminary Algorithm

In this section, we present a 2-local algorithm that constructs a graph, called the *plane localized Delaunay graph* $PLDG(V)$, whose vertex set is a finite set V of points in the plane. The algorithm computes $PLDG(V)$ in one communication round and each point of V broadcasts at most 6 messages. We will prove that $PLDG(V)$ is a plane and consistent supergraph of $UDel(V)$.

In the construction, each point v of V runs algorithm PLDG(v) in parallel. Let $N_v = N_1(v)$, i.e., $N_v = \{u \in V : |uv| \leq 1\}$. Recall that we assume that, at the start of the algorithm, point v knows the locations of all points in N_v. Algorithm PLDG(v) first computes the Delaunay triangulation $LDT(v)$ of the set N_v. Then, for each triangular face $\Delta(u, v, w)$ in $LDT(v)$ for which $\angle uvw > \frac{\pi}{3}$, algorithm PLDG($v$) broadcasts the location v together with the center of the disk $D(u, v, w)$ containing u, v, and w on its boundary.

In the final step, algorithm PLDG(v) checks the validity of all edges that are incident on v in $LDT(v)$ and removes those edges which cause a crossing. To be more precise, let x be a point in N_v, and assume that v receives a center c'_i from x. Algorithm PLDG(v) considers the unit-disk $D(v; 1)$ centered at v and the disk $D(c'_i; |c'_i x|)$ centered at c'_i that contains x on its boundary. The algorithm knows that $\partial D(c'_i; |c'_i x|)$ contains exactly three points which define a triangular face in the Delaunay triangulation $LDT(x)$ of N_x. Point x is one of these three points; let p and q be the other two points. Assume that the set N_v

contains exactly two points of $\{x, p, q\}$, say x and p. Thus, algorithm PLDG(v) knows the points x and p, but it does not know q. The algorithm computes arc_i, which is defined to be the (open) portion of $\partial D(c'_i; |c'_i x|)$ which is not contained in $D(v; 1)$. Even though the algorithm does not know the exact location of the third point q, it does know that q is on arc_i. The algorithm chooses an arbitrary point z' on arc_i such that $|xz'| \leq 1$ or $|pz'| \leq 1$ and *acts as if* $\Delta(x, p, z')$ is a triangular face in $LDT(x)$. (Observe that, since $q \in arc_i$ and $|xq| \leq 1$, the algorithm can choose such a point z'. Also, z' is not necessarily a point of V.) The algorithm now considers each edge (v, y) in $LDT(v)$ (where, possibly, $v = p$, $y = p$, or $y = x$) and uses the triangle $\Delta(x, p, z')$ to decide whether or not to remove (v, y): Since (v, y) is an edge in $LDT(v)$, algorithm PLDG(v) can compute a disk $D = Del_v(v, y)$ such that (i) v and y are the only points of N_v that are on the boundary of D and (ii) the interior of D does not contain any point of N_v. If arc_i is fully contained in the interior of $Del_v(v, y)$, then the algorithm knows that q is contained in the interior of $Del_v(v, y)$ (even though it does not know the exact location of q) and, therefore, $Del_v(v, y)$ is not a certificate that (v, y) is an edge in the Delaunay triangulation of the entire set V. Therefore, the algorithm checks if (i) arc_i is fully contained in the interior of $Del_v(v, y)$ and (ii) the line segment vy crosses any of the two line segments xz' and pz'. If both (i) and (ii) hold, the algorithm removes the edge (v, y). Observe that if (v, y) is not an edge of the Delaunay triangulation $DT(V)$, the algorithm still keeps it as long as it does not cross any other edge.

The formal algorithm is given in Figure 1. An illustration, with the special cases when $v = p$, $y = p$, or $y = x$, is given in Figure 2.

Remark 1 In the algorithm of Araujo and Rodrigues [1], for each triangular face $\Delta(u, v, w)$ in $LDT(v)$ for which $\angle uvw > \frac{\pi}{3}$, node v broadcasts the three points u, v, and w. Each node that receives these points uses them to remove certain edges from its local graph; see [1] for the details on how it is decided which edges to remove. Since there are at most five such triangles $\Delta(u, v, w)$, and v is common to all to them, node v broadcasts at most 11 messages. Our algorithm improves on this by only broadcasting the center of the circumcircle of $\Delta(u, v, w)$. Because of this, the "clean-up" step (i.e., the for-loop in lines 9–19) in our algorithm is very different from the one in [1].

Running algorithm PLDG(v) for all points v of V in parallel will be referred to as running algorithm PLDG(V). We denote by $E(v)$ the edge set that is computed by algorithm PLDG(v). Observe that each edge in $E(v)$ is incident on the point v. Let $E = \cup_{v \in V} E(v)$ and let $PLDG(V)$ denote the graph with vertex set V and edge set E.

In the rest of this section, we will prove a sequence of lemmas which lead to the proof that $PLDG(V)$ is a plane and consistent supergraph of $UDel(V)$; see Lemmas 5, 7 and 8.

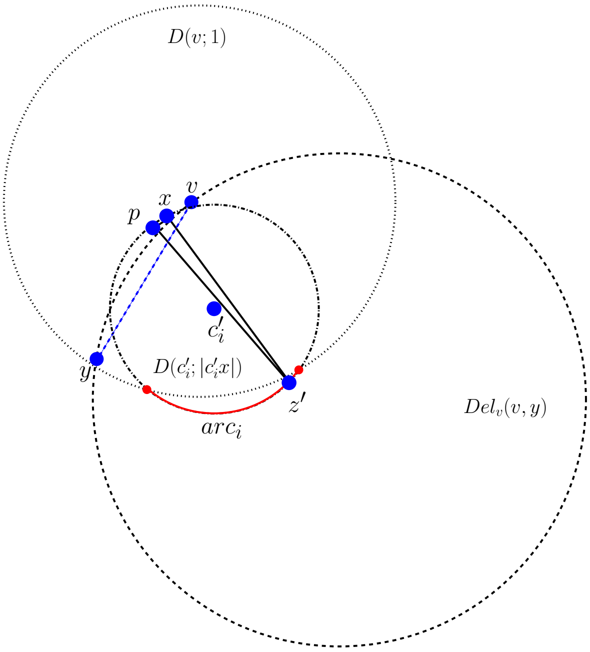
We start with a well known, but fundamental lemma:

Lemma 1 *Let $S = \{u, v, w, z\}$ be a set of four points in the plane in general position, such that $|uv| \leq 1$, $|wz| \leq 1$, and the line segments uv and wz cross. Then there exists a point x in S such that $|xy| \leq 1$ for all y in S.*

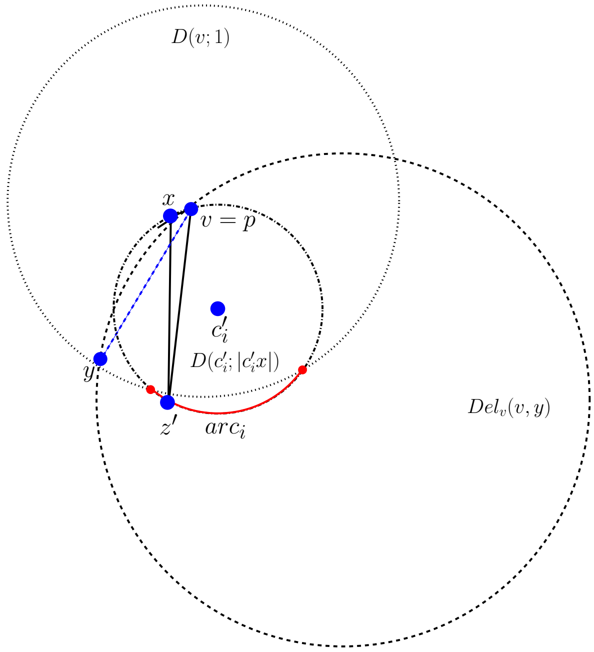
Algorithm PLDG(v)

let $N_v = \{u \in V : |uv| \leq 1\}$;
compute the Delaunay triangulation $LDT(v)$ of N_v;
let $E(v)$ be the set of all edges in $LDT(v)$ that are incident on v;
let Δ_v be the set of all triangular faces $\Delta(u, v, w)$ in $LDT(v)$ for which $\angle uvw > \frac{\pi}{3}$;
let k be the number of elements in Δ_v;
if $k \geq 1$
  then let $c_1, \ldots, c_k$ be the centers of the circumcircles of all triangles in Δ_v;
    broadcast the sequence $(v, c_1, \ldots, c_k)$;
for each sequence $(x, c'_1, \ldots, c'_m)$ received
  do for $i = 1$ **to** m
    do let $D(c'_i; |c'_i x|)$ be the disk with center c'_i that contains x on its boundary;
      if $\partial D(c'_i; |c'_i x|)$ contains exactly two points of N_v
        then let p be the point in $(N_v \setminus \{x\}) \cap \partial D(c'_i; |c'_i x|)$;
          let arc_i be the (open) arc on $\partial D(c'_i; |c'_i x|)$ that is not contained in the unit-disk $D(v; 1)$ centered at v;
          let z' be an arbitrary point on arc_i with $|xz'| \leq 1$ or $|pz'| \leq 1$;
          for each edge (v, y) in $E(v)$
            do let $Del_v(v, y)$ be a disk D such that $N_v \cap \partial D = \{v, y\}$ and $N_v \cap int(D) = \emptyset$;
              if arc_i is contained in the interior of $Del_v(v, y)$ and the line segment vy crosses at least one of the line segments xz' and pz'
                then remove (v, y) from $E(v)$

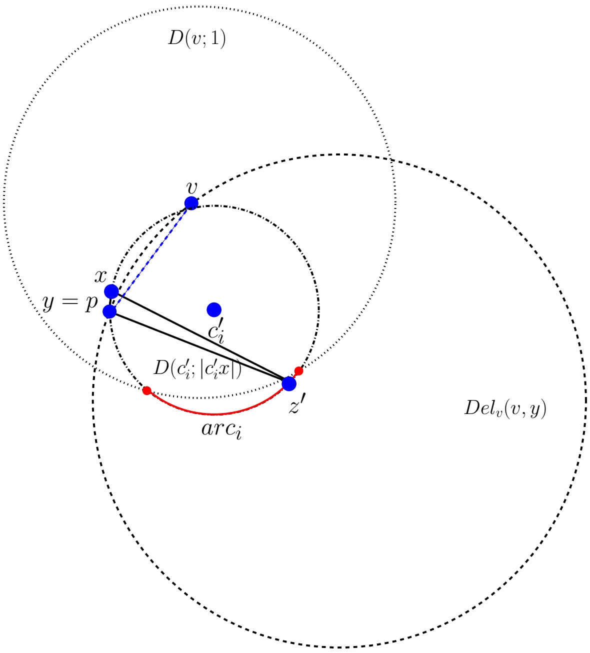
Figure 1: *The plane localized Delaunay graph algorithm.*



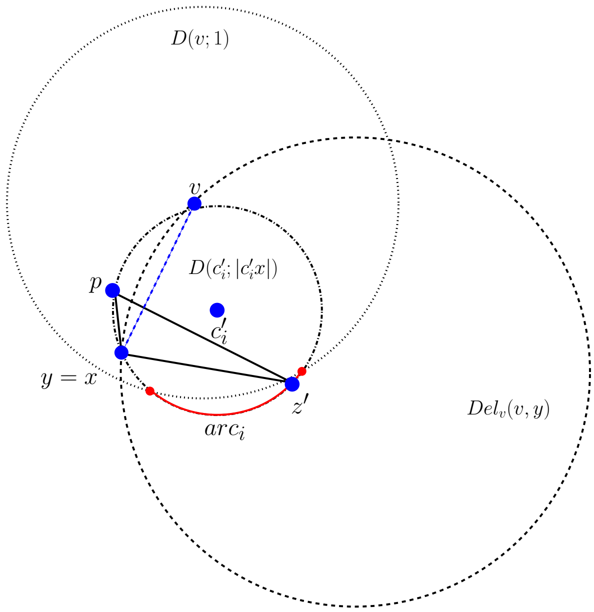
(a) edge (v, y) is removed, where $y \neq x$, $y \neq p$, and $v \neq p$.



(b) edge (v, y) is removed, where $v = p$.



(c) edge (v, y) is removed, where $y = p$.



(d) edge (v, y) is removed, where $y = x$.

Figure 2: *Illustrating algorithm* PLDG(v).

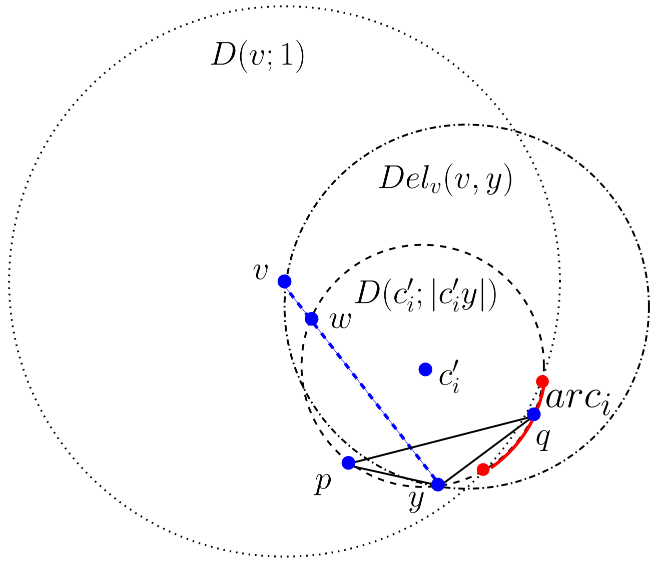


Figure 3: *An illustration of the proof of Lemma 2.*

The following lemma implies that for every edge (v, y) in $E(v)$, the edge (v, y) is in the Delaunay triangulation $LDT(y)$ of the set N_y.

Lemma 2 *Let v and y be two distinct points of V and assume that (v, y) is not an edge in $LDT(y)$. Then, after algorithm* PLDG(V) *has terminated, (v, y) is not an edge in $E(v)$.*

Proof. First assume that (v, y) is not an edge in $LDT(v)$. Then, since $E(v)$ is a subset of the edge set of $LDT(v)$, (v, y) is not an edge in $E(v)$.

From now on, we assume that (v, y) is an edge in $LDT(v)$. Observe that $|vy| \leq 1$. Since (v, y) is not an edge in $LDT(y)$, there exist two points p and q in V such that the triangle $\Delta(y, p, q)$ is a triangular face in $LDT(y)$ and vy crosses pq; see Figure 3. Observe that the points p, q, v, and y are pairwise distinct. In the rest of the proof, we will do the following:

1. We first show that algorithm PLDG(y) broadcasts the center of the circumcircle of $\Delta(y, p, q)$. Since $|vy| \leq 1$, v will receive this center.

2. We then show that, when algorithm PLDG(v) considers the center of $\Delta(y, p, q)$, it deletes the edge (v, y). As a result, the edge (v, y) is not in $E(v)$.

Let c_i' be the center of the circumcircle of $\Delta(y, p, q)$ and consider the corresponding disk $D(c_i'; |c_i'y|)$, i.e., the disk with center c_i' that contains y, p, and q on its boundary. Recall that $D(v; 1)$ denotes the unit-disk centered at v and $N_v = \{u \in V : |uv| \leq 1\}$.

Since $|vy| \leq 1$ and $\Delta(y, p, q)$ is a triangular face in $LDT(y)$, v is not contained in $D(c_i'; |c_i'y|)$. Since vy crosses pq, this implies that any disk D with v and y on its boundary contains at least one of p and q (otherwise, ∂D and $\partial D(c_i'; |c_i'y|)$ intersect more than twice).

We first claim that $\partial D(c_i'; |c_i'y|)$ contains exactly two points of N_v. Since $y \in N_v$, this means that we claim that exactly one of p and q is in N_v. We prove this by contradiction. First assume that neither p nor q is in N_v. Then both p and q are outside $D(v; 1)$. Let D be the disk with diameter vy. Since $|vy| \leq 1$, D is contained in $D(v; 1)$. Thus, neither p nor q is contained in D, which is a contradiction, because D contains v and y on its boundary. Now assume that both p and q are in N_v. Then, since any disk with v and y on its boundary contains one of p and q, it follows that (v, y) is not an edge in $LDT(v)$, which is again a contradiction.

Thus, we have shown that $\partial D(c_i'; |c_i'y|)$ contains exactly two points of N_v. We may assume without loss of generality that $y, p \in N_v$ and $q \notin N_v$.

Consider the triangle $\Delta(v, y, q)$. Since $|yv| \leq 1$, $|yq| \leq 1$, and $|vq| > 1$, we have $\angle vyq > \frac{\pi}{3}$. Since $\angle pyq > \angle vyq$, it follows that $\angle pyq > \frac{\pi}{3}$. Since $\Delta(y, p, q)$ is a triangular face in $LDT(y)$, algorithm PLDG(y) broadcasts a sequence in line 8 which contains the center c_i' of $D(c_i'; |c_i'y|)$.

As we have mentioned above, since $|vy| \leq 1$, v receives the sequence broadcast by PLDG(y). This sequence contains the center c_i' together with the point y. When algorithm PLDG(v) considers c_i', it discovers that $\partial D(c_i'; |c_i'y|)$ contains exactly two points of N_v; as we have seen above, these points are y and p. Thus, the condition in line 12 is satisfied. In line 14, algorithm PLDG(v) computes the open arc arc_i, which is the part of $\partial D(c_i'; |c_i'y|)$ that is not contained in $D(v; 1)$. Observe that even though PLDG(v) does not know the location of the point q, the algorithm knows that it is on arc_i. Let $Del_v(v, y)$ be the disk that is computed by PLDG(v) in line 17. This disk has the properties that $N_v \cap \partial Del_v(v, y) = \{v, y\}$ and $N_v \cap int(Del_v(v, y)) = \emptyset$.

We show that arc_i is contained in the interior of $Del_v(v, y)$; thus, the first condition in line 18 is satisfied. Let w be the intersection between vy and $\partial D(c_i'; |c_i'y|)$, let $\widehat{wpy}$ be the arc on $\partial D(c_i'; |c_i'y|)$ with endpoints w and y and which contains p, and let $\widehat{yqw}$ be the arc on $\partial D(c_i'; |c_i'y|)$ with endpoints y and w and which contains q. Since $|vy| \leq 1$, $|vq| > 1$, and $q \in \widehat{yqw}$, we have $\widehat{wpy} \subseteq D(v; 1)$ (because otherwise, $\partial D(v; 1)$ and $\partial D(c_i'; |c_i'y|)$ intersect more than twice). It follows that $arc_i \subseteq \widehat{yqw}$. Since $|vp| \leq 1$, we have $p \notin Del_v(v, y)$. Therefore, $\partial Del_v(v, y)$ and $\widehat{wpy}$ intersect twice. Since $\partial Del_v(v, y)$ and $\partial D(c_i'; |c_i'y|)$ cannot intersect more than twice, it follows that arc_i is contained in the interior of $Del_v(v, y)$.

Consider the point z' on arc_i that is chosen in line 15 of algorithm PLDG(v). We will show that vy crosses pz'; thus, the second condition in line 18 is also satisfied. Assume, by contradiction, that vy does not cross pz'. Since the line through v and y separates p from the two points q and z', and since vy crosses pq, it follows that y or v is in the triangle $\Delta(p, q, z')$. However, since y, p, q, and z' are on the circle $\partial D(c_i'; |c_i'y|)$, y cannot be in $\Delta(p, q, z')$. Also, since $\Delta(p, q, z')$ is contained in $D(c_i'; |c_i'y|)$ and since $v \in N_y$, v cannot be in $\Delta(p, q, z')$, because otherwise, $\Delta(y, p, q)$ would not be a triangular face in $LDT(y)$. Thus, we have shown that vy crosses pz'.

By inspecting algorithm PLDG(v), it follows that it removes, in line 19, the edge (v, y) from the edge set $E(v)$. This completes the proof. ∎

The following simple geometric lemma is stated without proof.

Lemma 3 *Let p and q be two points with $|pq| \leq 1$, let D be a disk containing p and q on its boundary, and let D_{cap} be the part of D that is bounded by the line segment pq and the minor arc $\widehat{pq}$ on ∂D between p and q. Then $|xy| \leq 1$ for all x and y in D_{cap}.*

The next lemma will form the basis for our claim that the graph $PLDG(V)$ is plane.

Lemma 4 *Let x, q, v, and y be four pairwise distinct points of V. Assume that $|xq| \leq 1$, $|xv| \leq 1$, $|xy| \leq 1$, $|vy| \leq 1$, xq crosses vy, (x, q) is an edge in $LDT(x)$, and (v, y) is an edge in $LDT(y)$. Then, after algorithm* PLDG(V) *has terminated, (v, y) is not an edge in $E(y)$.*

Proof. If (v, y) is not an edge in $LDT(v)$, then the claim follows from Lemma 2. In the rest of the proof, we assume that (v, y) is an edge in $LDT(v)$. We have to show that algorithm PLDG(y) removes the edge (v, y) from $E(y)$. Thus, we have to show that there exists a point x' in N_y which broadcasts the center of the circumcircle of some triangular face in $LDT(x')$ and, based on this information, PLDG(y) removes (v, y). We will use the edge (x, q) to prove that such a point x' exists. We assume, without loss of generality, that vy is horizontal and v is to the right of y. For each $x' \in V \setminus \{v, y\}$, let

$$Q_{vy}(x') = \{q' \in V \setminus \{v, y\} : (x', q') \text{ is an edge in } LDT(x') \text{ and } x'q' \text{ crosses } vy\}.$$

We define

$$X_{vy} = \{x' \in V \setminus \{v, y\} : |x'y| \leq 1, |x'v| \leq 1, Q_{vy}(x') \neq \emptyset\}.$$

Since $q \in Q_{vy}(x)$, we have $Q_{vy}(x) \neq \emptyset$. Since $|xy| \leq 1$ and $|xv| \leq 1$, we have $x \in X_{vy}$ and, therefore, $X_{vy} \neq \emptyset$.

Let x' be the leftmost point in X_{vy}. Let q' be the point in $Q_{vy}(x')$ such that the intersection between $x'q'$ and vy is closest to y. We assume, without loss of generality, that x' is above the line through vy. Since $x'q'$ crosses vy, the point q' is below the line through vy. Observe that x', q', v, and y are pairwise distinct.

By definition, (x', q') is an edge in $LDT(x')$. Let p' be the point of V such that $\Delta(x', p', q')$ is a triangular face in $LDT(x')$ and p' is to the left of the directed line from q' to x'; refer to Figure 4. Since $y \in N_{x'}$ and y is to the left of this line, the point p' exists. Observe that p' may be equal to y.

The following two facts imply that (i) p' is not below the line through vy, and (ii) in the case when $p' \neq y$, $p'q'$ crosses vy: First, since $y \in N_{x'}$ and $\Delta(x', p', q')$ is a triangular face in $LDT(x')$, the point y cannot be in $\Delta(x', p', q')$. Second, by our choice of q', the line segments $x'p'$ and vy do not cross.

In the rest of the proof, we will prove the following two claims:

1. Algorithm PLDG(x') broadcasts the center of the circumcircle of $\Delta(x', p', q')$. Since $|x'y| \leq 1$, y will receive this center.

2. When algorithm PLDG(y) considers the center of the circumcircle of $\Delta(x', p', q')$, it deletes the edge (v, y). As a result, the edge (v, y) is not in $E(y)$.

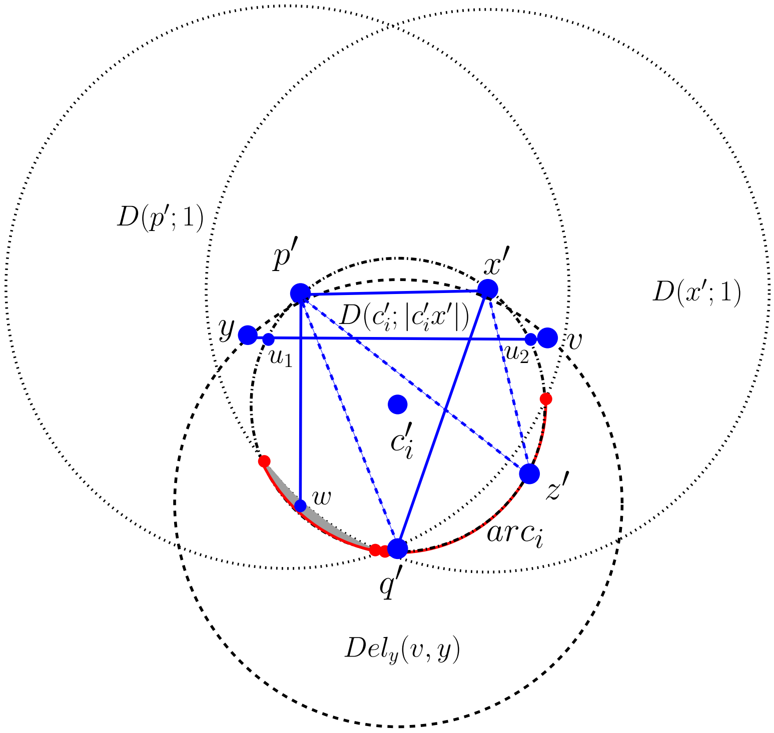


Figure 4: *An illustration of the proof of Lemma 4.*

Let c_i' be the center of the circumcircle of $\Delta(x',p',q')$ and consider the corresponding disk $D(c_i';|c_i'x'|)$. Recall that $D(x';1)$ denotes the unit-disk centered at x'.

Since $|x'y|\leq 1$, $|x'v|\leq 1$, and $\Delta(x',p',q')$ is a triangular face in $LDT(x')$, neither v nor y is contained in the interior of $D(c_i';|c_i'x'|)$. Moreover, since $v\notin\{x',p',q'\}$, v is not contained in $\partial D(c_i';|c_i'x'|)$. Finally, in the case when $y\neq p'$, y is not contained in $\partial D(c_i';|c_i'x'|)$. Since vy crosses $x'q'$, it follows that any disk D with v and y on its boundary contains at least one of x' and q' (because otherwise, ∂D and $\partial D(c_i';|c_i'x'|)$ intersect more than twice).

We now show that $|yq'|>1$. Assume, by contradiction, that $|yq'|\leq 1$. Since (v,y) is an edge in $LDT(y)$, there exists a disk $Del_y(v,y)$ having the property that $N_y\cap\partial Del_y(v,y)=\{v,y\}$ and $N_y\cap int(Del_y(v,y))=\emptyset$. Since both x' and q' are in N_y, neither of these two points is contained in $Del_y(v,y)$, which is a contradiction. Since (v,y) is an edge in $LDT(v)$, a symmetric argument implies that $|vq'|>1$.

Consider the triangle $\Delta(v,y,q')$. Since $|vq'|>1$, $|yq'|>1$, and $|vy|\leq 1$, we have $\angle vq'y<\frac{\pi}{3}$. Since $\angle x'q'p'\leq\angle vq'y$ it follows that $\angle x'q'p'<\frac{\pi}{3}$. Next, consider the triangle $\Delta(x',p',q')$. Since $\angle x'p'q'+\angle p'x'q'>\frac{2\pi}{3}$, at least one of $\angle x'p'q'$ and $\angle p'x'q'$ is larger than $\frac{\pi}{3}$. Below, we will prove that $\angle p'x'q'>\frac{\pi}{3}$. Since $\Delta(x',p',q')$ is a triangular face in $LDT(x')$, this will imply that algorithm PLDG(x') broadcasts a sequence in line 8 which contains the center c_i' of $D(c_i';|c_i'x'|)$.

Assume, by contradiction, that $\angle p'x'q'\leq\frac{\pi}{3}$. Then $\angle x'p'q'>\frac{\pi}{3}$. Since $|x'p'|\leq 1$, $|x'q'|\leq 1$ and $\angle p'x'q'\leq\frac{\pi}{3}$, we have $|p'q'|\leq 1$. We first prove, again by contradiction, that

$\Delta(x', p', q')$ is not a triangular face in $LDT(p')$. Thus, we assume that it is a triangular face in $LDT(p')$. Since vy crosses $x'q'$, and (v, y) is an edge in $LDT(y)$, we have $p' \neq y$ (because otherwise, $LDT(p')$ would not be plane). Refering again to Figure 4, let u_1 and u_2 be the two intersection points between $D(c'_i; |c'_i x'|)$ and vy, where u_1 is to the left of u_2. We have $\angle u_1 q' u_2 \leq \angle v q' y < \frac{\pi}{3}$. Consider the arc $\widehat{u_1 x' u_2}$ on $\partial D(c'_i; |c'_i x'|)$ with endpoints u_1 and u_2 that contains x'. This arc is a minor arc on $\partial D(c'_i; |c'_i x'|)$. Since $p' \in \widehat{u_1 x' u_2}$ and p' is to the left of the line through $x'q'$, it follows that p' is to the left of x'. Then, by our choice of x', we have $p' \notin X_{vy}$. Thus, by the definition of X_{vy}, we have (i) $|p'y| > 1$ or (ii) $|p'v| > 1$ or (iii) $Q_{vy}(p') = \emptyset$. Since $q' \in Q_{vy}(p')$, (iii) does not hold. Since vy and $p'q'$ cross, $|vy| \leq 1$, $|p'q'| \leq 1$, $|yq'| > 1$, and $|vq'| > 1$, it follows from Lemma 1 that $|p'y| \leq 1$ and $|p'v| \leq 1$; thus, neither (i) nor (ii) holds, which is a contradiction. We conclude that $\Delta(x', p', q')$ is not a triangular face in $LDT(p')$.

We continue deriving a contradiction to the assumption that $\angle p' x' q' \leq \frac{\pi}{3}$. Since $\Delta(x', p', q')$ is a triangular face in $LDT(x')$ but not in $LDT(p')$, there exists at least one point w of V in the interior of $D(c'_i; |c'_i x'|)$ such that $|x'w| > 1$ and $|p'w| \leq 1$. Let W be the set of all such points w, i.e.,

$$W = \{w \in V : w \in int(D(c'_i; |c'_i x|)), |x'w| > 1, |p'w| \leq 1\}.$$

For each $w \in W$, let R_w be the radius of the circle through p' and w and whose center is on $p'c'_i$. Let w be a point in W for which R_w is minimum. Let D_w be the disk centered on $p'c'_i$ that contains p' and w on its boundary. Observe that D_w is contained in $D(c'_i; |c'_i x'|)$. Also, no point of W is in the interior of D_w. It follows that (p', w) is an edge in $LDT(p')$.

We have seen above that p' is to the left of x'. It follows that $p' \notin X_{vy}$. Thus, by the definition of X_{vy}, we have (i) $|p'y| > 1$, or (ii) $|p'v| > 1$, or (iii) $Q_{vy}(p') = \emptyset$, or (iv) $p' = y$. The arguments above show that neither (i) nor (ii) holds. Assume that (iv) does not hold, i.e., $p' \neq y$. We show that $w \in Q_{vy}(p')$; this will imply that (iii) does not hold. Since $w \in int(D(c'_i; |c'_i x|))$, we have $w \neq v$ and $w \neq y$. Thus, in order to show that $w \in Q_{vy}(p')$, it suffices to show that $p'w$ crosses vy. Consider again the two intersection points u_1 and u_2 between $D(c'_i; |c'_i x'|)$ and vy, where u_1 is to the left of u_2. As we have seen before, the arc $\widehat{u_1 x' u_2}$ on $\partial D(c'_i; |c'_i x'|)$ is a minor arc. Since $|u_1 u_2| \leq |vy| \leq 1$, it follows from Lemma 3 that w is below the line through v and y (because otherwise, we would have $|wx'| \leq 1$, contradicting the fact that $w \in W$). Thus, since p' and w are on opposite sides of the line through v and y, and since $w \in D(c'_i; |c'_i x'|)$, this shows that $p'w$ crosses vy. As mentioned above, this implies that (iii) does not hold. We conclude that (iv) holds, i.e., $p' = y$. In the triangle $\Delta(p', x', q')$, we have $|p'x'| \leq 1$, $|x'q'| \leq 1$, and $|p'q'| = |yq'| > 1$. It follows that $\angle p' x' q' > \frac{\pi}{3}$, which is a contradiction.

Thus, we have obtained a contradiction to the assumption that $\angle p' x' q' \leq \frac{\pi}{3}$. As a result, we conclude that $\angle p' x' q' > \frac{\pi}{3}$. As we mentioned before, this implies that algorithm PLDG(x') broadcasts a sequence in line 8 which contains the center c'_i of $D(c'_i; |c'_i x'|)$ (which is the circumscribing disk of the triangular face $\Delta(x', p', q')$ in $LDT(x')$).

Since $|yx'| \leq 1$, y receives the sequence broadcast by algorithm PLDG(x'). This sequence contains the center c'_i together with the point x'. Recall that $|yq'| > 1$. Let D be the disk whose boundary contains y, v, and the "north pole" of $D(c_i'; |c_i'x'|)$. Since $\angle vq'y < \frac{\pi}{3}$, the center of D is below the line through v and y. Since $|vy| \leq 1$, it then follows from Lemma 3 that $|yp'| \leq 1$. Thus, when algorithm PLDG(y) considers c_i', it discovers that the boundary of the disk $D(c_i'; |c_i'x'|)$ contains exactly two points of N_y; these are the points x' and p'. Algorithm PLDG(y) computes the open arc arc_i, which is the part of $\partial D(c_i'; |c_i'x'|)$ that is not contained in the unit-disk $D(y; 1)$ centered at y. The algorithm knows that the third point q' on $\partial D(c_i'; |c_i'x'|)$ is somewhere on arc_i. Let $Del_y(v, y)$ be the disk that is computed in line 17 of algorithm PLDG(y). This disk has the properties that $N_y \cap \partial Del_y(v, y) = \{v, y\}$ and $N_y \cap int(Del_y(v, y)) = \emptyset$. By the same argument as in the proof of Lemma 2, arc_i is contained in the interior of $Del_y(v, y)$. Moreover, arc_i is below the line through v and y.

Consider the point z' on arc_i that is chosen in line 15 of algorithm PLDG(y). We will show that vy crosses $x'z'$. Assume, by contradiction, that vy does not cross $x'z'$. Since the line through v and y separates x' from the two points q' and z', and since vy crosses $x'q'$, it follows that v or y is in the triangle $\Delta(x', q', z')$. Thus, v or y is in the interior of the disk $D(c_i'; |c_i'x'|)$. This is a contradiction, because $|x'v| \leq 1$, $|x'y| \leq 1$, and $\Delta(x', p', q')$ is a triangular face in $LDT(x')$. Thus, we have shown that vy crosses $x'z'$.

It now follows from the description of the algorithm that PLDG(y) removes the edge (v, y) from $E(y)$. This completes the proof of the lemma. ■

Based on the previous lemmas, we can now prove that $PLDG(V)$ is plane:

Lemma 5 *$PLDG(V)$ is a plane graph.*

Proof. The proof is by contradiction. Assume that $PLDG(V)$ contains two crossing edges (v, y) and (x, q). By Lemma 1, one of the points in $\{x, q, v, y\}$ is within distance 1 from the other three points. We may assume without loss of generality that $|xq| \leq 1$, $|xv| \leq 1$, and $|xy| \leq 1$. By Lemma 2, (v, y) is an edge in $LDT(v)$ and in $LDT(y)$, and (x, q) is an edge in $LDT(x)$.

Since all conditions in Lemma 4 are satisfied, (v, y) is not an edge in $E(y)$. Also, the conditions in Lemma 4, with v and y interchanged, are satisfied. Therefore, (v, y) is not an edge in $E(v)$. Thus, (v, y) is not an edge in $PLDG(V)$, which is a contradiction. ■

The following lemma summarizes the different scenarios when algorithm PLDG(v) removes an edge (v, y) from the edge set $E(v)$.

Lemma 6 *Let v and y be two distinct points of V such that (v, y) is an edge in $LDT(v)$. Assume that algorithm* PLDG(v) *removes (v, y) from $E(v)$. Then, there exist three pairwise distinct points x, p, and q in V such that*

1. *$\Delta(x, p, q)$ is a triangular face in $LDT(x)$,*
2. *$v \neq x$, $|vx| \leq 1$, $|vp| \leq 1$, $|vq| > 1$,*
3. *neither v nor y is in the interior of the disk $D(x, p, q)$, and*

4. (a) *if* $y \neq x$*,* $v \neq p$*, and* $y \neq p$*, the line segment* vy *crosses both the line segments* xq *and* pq*,*

   (b) *if* $y = x$*, the line segment* vy *crosses the line segment* pq*,*

   (c) *if* $v = p$*, the line segment* vy *crosses the line segment* xq*,*

   (d) *if* $y = p$*, the line segment* vy *crosses the line segment* xq*.*

Proof. Since algorithm PLDG(v) removes (v, y) from $E(v)$, there exists a point x in $N_v \setminus \{v\}$ which broadcasts the center c'_i of the circumcircle of a triangular face $\Delta(x, p, q)$ in $LDT(x)$, such that the following holds:

1. Consider the disk $D(c'_i; |c'_i x|) = D(x, p, q)$ with center c'_i that contains x, p, and q on its boundary. Then, according to line 12 of algorithm PLDG(v), $\partial D(c'_i; |c'_i x|)$ contains exactly two points of N_v. Since we assume that no four points of V are cocircular, x, p, and q are the only points of V that are on $\partial D(c'_i; |c'_i x|)$. Thus, since $x \in N_v$, exactly one of p and q is in N_v. We may assume without loss of generality that $p \in N_v$ and $q \notin N_v$.

2. Consider the unit-disk $D(v; 1)$ centered at v. Let arc_i be the arc on $\partial D(c'_i; |c'_i x|)$ that is not contained in $D(v; 1)$, let z' be the point on arc_i with $|xz'| \leq 1$ or $|pz'| \leq 1$ that is chosen by algorithm PLDG(v) in line 15, and let $Del_v(v, y)$ be the disk chosen in line 17. Thus, v and y are the only points of N_v that are on $\partial Del_v(v, y)$ and no point of N_v is in the interior of $Del_v(v, y)$. Then, by line 18 of algorithm PLDG(v), arc_i is contained in the interior of $Del_v(v, y)$ and vy crosses at least one of xz' and pz'.

The first two claims in the lemma hold for the points x, p, and q.

We now prove the third claim. Since $|xv| \leq 1$ and $\Delta(x, p, q)$ is a triangular face in $LDT(x)$, v is not in the interior of the disk $D(x, p, q)$. We prove by contradiction that y is not in the interior of $D(x, p, q)$. Thus, we assume that y is in the interior of this disk. Then, again since $\Delta(x, p, q)$ is a triangular face in $LDT(x)$, we have $|xy| > 1$. Recall that (i) $|xz'| \leq 1$ or $|pz'| \leq 1$ and (ii) vy crosses at least one of xz' and pz'. Therefore, we distinguish four cases and derive a contradiction for each of them.

Case 1: $|xz'| \leq 1$ and vy crosses xz'.

Since $|vy| \leq 1$ and $|vz'| > 1$, Lemma 1 implies that $|xy| \leq 1$, which is a contradiction.

Case 2: $|xz'| \leq 1$ and vy does not cross xz'.

In this case, vy crosses pz'. Observe that $x \neq y$, $v \neq p$, and $y \neq p$. Also, $v \notin D(x, p, q)$. The following observations lead to a contradiction:

- Since $|xp| \leq 1$ and $|xz'| \leq 1$, each point in the triangle $\Delta(x, p, z')$ has distance at most one to x. Therefore, $y \notin \Delta(x, p, z')$.

- The line segment vy crosses px. This follows from the facts that vy does not cross xz', vy crosses pz', $v \notin D(x, p, q)$, $y \in int(D(x, p, q))$, and $y \notin \Delta(x, p, z')$.

- The line segment px is disjoint from arc_i: Since $|vp| \leq 1$ and $|vx| \leq 1$, each point on px has distance at most one to v. Thus, $px \subseteq D(v; 1)$. However, arc_i and $D(v; 1)$ are disjoint.

- arc_i and v are on the same side of the line through p and x: Assume this is not the case. Since neither p nor x is in $Del_v(v, y)$ and since arc_i is in the interior of $Del_v(v, y)$, it follows that $\partial Del_v(v, y)$ and $\partial D(x, p, q)$ intersect more than twice. This is a contradiction.

- Let $\widehat{px}$ be the arc on $\partial D(x, p, q)$ between p and x that does not contain z'. We claim that $\widehat{px}$ is a major arc. To prove this, assume that it is a minor arc. The observations above imply that y is in the region of $D(x, p, q)$ that is bounded by px and $\widehat{px}$. Since $|xp| \leq 1$, it then follows from Lemma 3 that $|xy| \leq 1$, which is a contradiction.

- Let v' be the intersection between xv and $\partial D(x, p, q)$. Let $\widehat{pv'x}$ be the arc on $\partial D(x, p, q)$ between p and x that contains v'. Since $|vp| \leq 1$, $|vx| \leq 1$, and $\widehat{pv'x}$ is a minor arc, we know that $\widehat{pv'x}$ is contained in $D(v; 1)$. Since q is on $\widehat{pv'x}$, it follows that $|vq| \leq 1$, which is a contradiction.

Case 3: $|xz'| > 1$ and vy crosses xz'.
The following observations lead to a contradiction:

- Since $|vy| \leq 1$, $|xq| \leq 1$, $|xy| > 1$, and $|vq| > 1$, it follows from Lemma 1 that vy and xq do not cross. This also implies that $q \neq z'$.

- The points q and z' are on the same side of the line through v and y: This follows from the facts that both q and z' are on arc_i, arc_i is contained in the interior of $Del_v(v, y)$, $arc_i \cap D(v; 1) = \emptyset$, and $|vy| \leq 1$.

- Since vy crosses xz' but not xq, and since q and z' are on the same side of the line through v and y, it follows that y is in the triangle $\Delta(x, q, z')$.

- Consider the unit-disk $D(x; 1)$ centered at x. Assume, without loss of generality that xq is vertical, q is above x, and z' is to the right of x. Observe that both v and q are contained in $D(x; 1)$, and neither y nor z' is contained in $D(x; 1)$. Since vy crosses xz' and $y \in \Delta(x, q, z')$, the point v is below the line through x and z. This implies that the line through v and y separates q and z', which is a contradiction.

Case 4: $|xz'| > 1$ and vy does not cross xz'.
In this case, vy crosses pz'. The following observations lead to a contradiction:

- The line segments vy and px do not cross: If they do cross, then the same analysis as in Case 2 leads to a contradiction.

- As in Case 3, the line segments vy and qx do not cross.

- Since vy crosses pz', but vy neither crosses xz' nor xp, and since $y \in int(D(x,p,q)$ and $v \notin int(D(x,p,q))$, the point y is in the triangle $\Delta(x,p,z')$.
- Since $|xp| \le 1$ and $|xq| \le 1$, each point in the triangle $\Delta(x,p,q)$ has distance at most one to x. Therefore, since $|xy| > 1$, we have $y \notin \Delta(x,p,q)$. In particular, $q \neq z'$.
- As in Case 2, the line segment px is disjoint from arc_i. Thus, q and z' are on the same side of the line through p and x.
- Assume without loss of generality that px is horizontal, p is to the left of x, and both q and z' are above the line through p and x.
- Let $\widehat{pqx}$ be the arc on $\partial D(x,p,q)$ that is above px. If this arc is a minor arc, then, since $|px| \le 1$ and using Lemma 3, we have $|xz'| \le 1$, which is a contradiction. Thus, $\widehat{pqx}$ is a major arc.
- Assume that y is on or below px. Since the arc on $\partial D(x,p,q)$ that is below px is a minor arc, it follows from Lemma 3 that $|xy| \le 1$, which is a contradiction. Thus, y is above px.
- Assume that y is to the right of xq. Since y is contained in $\Delta(x,p,z')$, the point z' is on the arc on $\partial D(x,p,q)$ between x and q that is to the right of xq. Recall that vy crosses neither xz' nor xq. It follows that vy crosses qz', which is a contradiction, because q and z' are on the same side of the line through v and y.
- We conclude that y is to the left of xq. Since y is contained in $\Delta(x,p,z')$ but not in $\Delta(x,p,q)$, the point z' is on the arc on $\partial D(x,p,q)$ between p and q that is to the left of xq.
- Assume that v is above the line through x and y. Since vy crosses pz', y is contained in the triangle $\Delta(x,p,v)$. However, since $|xv| \le 1$ and $|xp| \le 1$, this implies that $|xy| \le 1$, which is a contradiction. Thus, v is below the line through x and y.
- Since both q and z' are above the line through v and y, y is contained in the triangle $\Delta(x,v,q)$. However, since $|xv| \le 1$ and $|xq| \le 1$, this implies that $|xy| \le 1$, which is a contradiction.

To conclude, in each of the four cases above, we have obtained a contradiction to the assumption that y is in the interior of $D(x,p,q)$. Therefore, we have proved the third claim in the lemma.

It remains to prove the fourth claim in the lemma. First assume that $y \neq x$, $v \neq p$, and $y \neq p$. We first show that x and p are on the same side of the line through v and y. Assume, by contradiction, that x and p are on opposite sides of this line. Since both x and p are in N_v, neither of these two points is contained in $Del_v(v,y)$. On the other hand, since $arc_i \subseteq int(Del_v(v,y))$ and $z' \in arc_i$, the point z' is in the interior of $Del_v(v,y)$. We also know that neither v nor y is contained in $D(c_i'; |c_i'x|) = D(x,p,q)$. Since $\partial D(x,p,q)$ contains the points x, p, and z', it follows that the boundaries of $Del_v(v,y)$ and $D(x,p,q)$ intersect more than twice. This is a contradiction.

Assume, without loss of generality, that vy is horizontal and both x and p are above the line through v and y. Then z' is below this line. Since $arc_i \cap D(v;1) = \emptyset$, it follows that the entire arc arc_i is below this line. In particular, q is below the line through v and y. Since neither v nor y is contained in $D(x,p,q)$, since vy intersects $\partial D(x,p,q)$ twice, and since vy separates q from x and p, it follows that vy crosses both the line segments xq and pq.

It remains to prove the special cases in the fourth claim. First assume that $y = x$. Since vy does not cross $xz' = yz'$, we know that vy crosses pz', which implies that $v \neq p$ and $y \neq p$. Since the line through v and y separates p from q and z', it follows that vy crosses pq.

Next assume that $v = p$. Since vy does not cross $pz' = vz'$, we know that vy crosses xz', which implies that $y \neq x$. Since q and z' are on the same side of the line through v and y, it follows that vy crosses xq.

Finally, assume that $y = p$. Since vy does not cross $pz' = yz'$, we know that vy crosses xz'. Since the line through v and y separates x from q and z', it follows that vy crosses xq. This completes the proof of the lemma. ∎

We can now prove that $PLDG(V)$ is consistent:

Lemma 7 *The graph $PLDG(V)$ is consistent: For any two distinct points v and y of V, (v,y) is an edge in $E(v)$ if and only if (v,y) is an edge in $E(y)$.*

Proof. The proof is by contradiction. Assume there is a pair (v,y) which is an edge in $E(y)$ but not in $E(v)$. Then (v,y) is an edge in $LDT(y)$ and, by Lemma 2, (v,y) is an edge in $LDT(v)$. Since (v,y) is not an edge in $E(v)$, it has been removed by algorithm PLDG(v). Thus, by Lemma 6, there exist three pairwise distinct points x, p, and q in V such that (i) $\Delta(x,p,q)$ is a triangular face in $LDT(x)$, (ii) $v \neq x$, $|vx| \leq 1$, $|vq| > 1$, and (iii) the line segment vy crosses at least one of the line segments pq and xq.

Assume that vy does not cross xq. Then vy crosses pq and, by the fourth claim in Lemma 6, $y = x$. Thus, since (v,y) is an edge in $LDT(y) = LDT(x)$ and using (i), it follows that $LDT(x)$ is not plane, which is a contradiction.

Thus, vy crosses xq. This implies that the points x, q, v, and y are pairwise distinct. It follows from (i) that (x,q) is an edge in $LDT(x)$ and $|xq| \leq 1$. Since $|vy| \leq 1$, $|xq| \leq 1$, $|vq| > 1$, and since vy crosses xq, it follows from Lemma 1 that $|xy| \leq 1$. Thus, all conditions in Lemma 4 are satisfied. As a result, algorithm PLDG(y) deletes the edge (v,y) from $E(y)$. This is a contradiction. ∎

Recall that $UDel(V)$ denotes the intersection of the Delaunay triangulation and the unit-disk graph of V. We next show that $PLDG(V)$ contains $UDel(V)$.

Lemma 8 *The graph $UDel(V)$ is a subgraph of $PLDG(V)$.*

Proof. Let (v,y) be an edge of $UDel(V)$. We will show that (v,y) is an edge in $E(v)$. By definition, $|vy| \leq 1$ and (v,y) is an edge in the Delaunay triangulation of V. Therefore, (v,y)

is also an edge in the Delaunay triangulation $LDT(v)$ of N_v and, thus, (v, y) is added to the edge set $E(v)$ in line 3 of algorithm PLDG(v). We have to show that algorithm PLDG(v) does not remove (v, y) in line 19.

Assume that (v, y) is removed in line 19 of algorithm PLDG(v). By Lemma 6, there exist three pairwise distinct points x, p, and q in V such that (i) neither v nor y is in the interior of the disk $D(x, p, q)$ and (ii) the line segment vy crosses at least one of the line segments xq and pq.

Assume that vy crosses xq. Then, the points v, y, x, and q are pairwise distinct. Observe that p may be equal to v or y. Let D be an arbitrary disk having v and y on its boundary, and assume that neither x nor q is contained in D. Then it follows from (i) and (ii) that the boundaries of D and $D(x, p, q)$ intersect more than twice, which is a contradiction. Thus, D contains at least one of x and q. Since D was arbitrary, this contradicts the fact that (v, y) is an edge in the Delaunay triangulation of V.

By a symmetric argument, the case when vy crosses pq also leads to a contradiction to the fact that (v, y) is an edge in the Delaunay triangulation of V. ∎

In the next lemma, we summarize the results obtained in this section. Recall that a message is defined to be the location of a point in the plane.

Lemma 9 *Let V be a finite set of points in the plane. The distributed algorithm* PLDG(v)*, where v ranges over all points in V, is a* 2*-local algorithm that computes a plane and consistent $\frac{4\pi\sqrt{3}}{9}$-spanner $PLDG(V)$ of the unit-disk graph of V. This algorithm makes one communication round and each point of V broadcasts at most* 6 *messages.*

Proof. Let v be a point of V. Lines 1–8 of algorithm PLDG(v) depend only on the points in N_v. Lines 9–19 depend only on information received from nodes x in N_v; this information was computed in lines 1–8 of algorithm PLDG(x) and, thus, depends only on the points in N_x. It follows that algorithm PLDG(v) is 2-local. Algorithm PLDG(v) broadcasts a sequence of messages only once, in line 8. Therefore, there is only one round of communication. In the Delaunay triangulation $LDT(v)$ of N_v, there are at most 5 triangular faces $\Delta(u, v, w)$ with $\angle uvw > \frac{\pi}{3}$. Therefore, the sequence that is broadcast in line 8 contains at most 6 points. Thus, PLDG(v) broadcasts at most 6 messages.

By Lemmas 5 and 7, the graph $PLDG(V)$ is plane and consistent. By Lemma 8, $PLDG(V)$ is a supergraph of $UDel(V)$. Since $UDel(V)$ is a $\frac{4\pi\sqrt{3}}{9}$-spanner of the unit-disk graph $UDG(V)$ of V (see Bose *et al.* [2]), the graph $PLDG(V)$ is a $\frac{4\pi\sqrt{3}}{9}$-spanner of $UDG(V)$. ∎

3 The Final Algorithm

We have seen that in algorithm PLDG, each point of V broadcasts at most 6 messages. In this section, we improve this upper bound to 5. We obtain this improvement, by making the following modification to the algorithm: The sequence $(v, c_1, \ldots, c_k)$ that is broadcast in

Algorithm PLDG$'(v)$

1. let $N_v = \{u \in V : |uv| \leq 1\}$;
2. compute the Delaunay triangulation $LDT(v)$ of N_v;
3. let $E'(v)$ be the set of all edges in $LDT(v)$ that are incident on v;
4. let Δ_v be the set of all triangular faces $\Delta(u, v, w)$ in $LDT(v)$ for which $\angle uvw > \frac{\pi}{3}$;
5. let k be the number of elements in Δ_v;
6. **if** $k \geq 1$
7.   **then** let $c_1, \ldots, c_k$ be the centers of the circumcircles of all triangles in Δ_v;
8.   broadcast the sequence $(c_1, \ldots, c_k)$;
9. **for** each sequence $(c'_1, \ldots, c'_m)$ received
10.   **do for** $i = 1$ **to** m
11.   **do** compute a point x' in $N_v \setminus \{v\}$ that is closest to c'_i;
12.   let $D(c'_i; |c'_i x'|)$ be the disk with center c'_i that contains x' on its boundary;
13.   **if** $\partial D(c'_i; |c'_i x'|)$ contains exactly two points of N_v
14.   **then** let p' be the point in $(N_v \setminus \{x'\}) \cap \partial D(c'_i; |c'_i x'|)$;
15.   let $arc_i = (\partial D(c'_i; |c'_i x'|)) \setminus D(v; 1)$;
16.   let $Z = \{z' \in arc_i : |x'z'| \leq 1$ or $|p'z'| \leq 1\}$;
17.   **if** $arc_i \neq \emptyset$ and $Z \neq \emptyset$
18.   **then** let z' be an arbitrary element of Z;
19.   **for** each edge (v, y) in $E'(v)$
20.   **do** let $Del_v(v, y)$ be a disk D such that $N_v \cap \partial D = \{v, y\}$ and $N_v \cap int(D) = \emptyset$;
21.   **if** arc_i is contained in the interior of $Del_v(v, y)$ and the line segment vy crosses at least one of the line segments $x'z'$ and $p'z'$
22.   **then** remove (v, y) from $E'(v)$

Figure 5: *The improved plane localized Delaunay graph algorithm.*

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line 8 of algorithm PLDG(v) contains the location of the sender v. In our new algorithm, point v sends only the sequence $(c_1, \ldots, c_k)$ of centers. Thus, any point that receives this sequence does not know that the sequence was broadcast by v. Assume that v receives a center c'_i from some node x in N_v. Since v does not know that c'_i was broadcast by x, line 11 in algorithm PLDG(v) has to be modified. In the new algorithm, v computes a point x' in $N_v \setminus \{v\}$ that is closest to c'_i and uses the disk $D(c'_i; |c'_i x'|)$ to decide whether or not to remove an edge (v, y).

The new algorithm, which we denote by PLDG$'$, is given in Figure 5. We denote by $E'(v)$ the edge set that is computed by algorithm PLDG$'(v)$. Let $E' = \cup_{v \in V} E'(v)$ and let $PLDG'(V)$ denote the graph with vertex set V and edge set E'.

Recall that $E(v)$ denotes the edge set that is computed by algorithm PLDG(v) and $PLDG(V)$ denotes the graph with vertex set V and edge set $\cup_{v \in V} E(v)$. We claim that

$PLDG(V) = PLDG'(V)$; thus, the new algorithm PLDG′ computes the same graph as algorithm PLDG. In order to prove this claim, it suffices to show that algorithm PLDG(v) removes an edge (v, y) from $E(v)$ if and only if algorithm PLDG′(v) removes the edge (v, y) from $E'(v)$. We will show this in the following two lemmas.

Lemma 10 *Let v be an element of V and let (v, y) be an edge of the Delaunay triangulation $LDT(v)$ of the set N_v. If algorithm* PLDG(v) *removes (v, y) from $E(v)$, then algorithm* PLDG′(v) *removes (v, y) from $E'(v)$.*

Proof. By Lemma 6, there exist three pairwise distinct points x, p, and q in V such that

1. $\Delta(x, p, q)$ is a triangular face in $LDT(x)$,
2. $v \neq x$, $|vx| \leq 1$, $|vp| \leq 1$, $|vq| > 1$,
3. neither v nor y is in the interior of the disk $D(x, p, q)$.

In fact, in algorithm PLDG(v), v receives from x the center c'_i of the disk $D(c'_i; |c'_i x|) = D(x, p, q)$. Since $|vx| \leq 1$, in algorithm PLDG′(v), v receives the center c'_i, but does not know that it was broadcast by x. Consider the point x' that is computed in line 11 of algorithm PLDG′(v). Thus, x' is a point in $N_v \setminus \{v\}$ that is closest to c'_i. Since $x \in N_v \setminus \{v\}$, we have $|c'_i x'| \leq |c'_i x|$. We claim that $|c'_i x'| = |c'_i x|$.

To prove this claim, assume, by contradiction, that $|c'_i x'| < |c'_i x|$. Then x' is in the interior of the disk $D(x, p, q)$. Since $\Delta(x, p, q)$ is a triangular face in $LDT(x)$, we have $|xx'| > 1$.

Consider the disk $Del_v(v, y)$ that is computed in line 17 of algorithm PLDG(v). Recall that $N_v \cap \partial Del_v(v, y) = \{v, y\}$ and $N_v \cap int(Del_v(v, y)) = \emptyset$. Since both x and x' are in N_v, neither of these two points is in the interior of $Del_v(v, y)$. It follows from line 18 of algorithm PLDG(v) that q is in the interior of $Del_v(v, y)$ (because $q \in arc_i$).

Assume, without loss of generality that vy is horizontal, v is to the right of y, and q is below the line through v and y; refer to Figure 6.

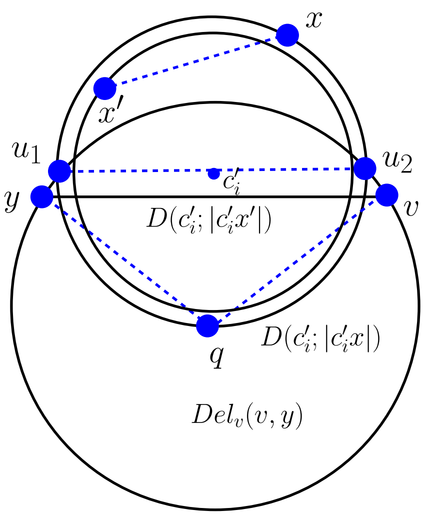
Since $|vq| > 1$ and $|vy| \leq 1$, we have $\angle yqv < \pi/2$. Let $\widehat{yv}$ be the arc on $\partial Del_v(v, y)$ with endpoints y and v that contains the north pole of $\partial Del_v(v, y)$. Then $\widehat{yv}$ is a minor arc.

Let u_1 and u_2 be the intersections between $\partial Del_v(v, y)$ and $\partial D(x, p, q)$, where u_1 is to the left of u_2. Then both u_1 and u_2 are contained in $\widehat{yv}$ and, therefore, by Lemma 3, $|u_1 u_2| \leq 1$.

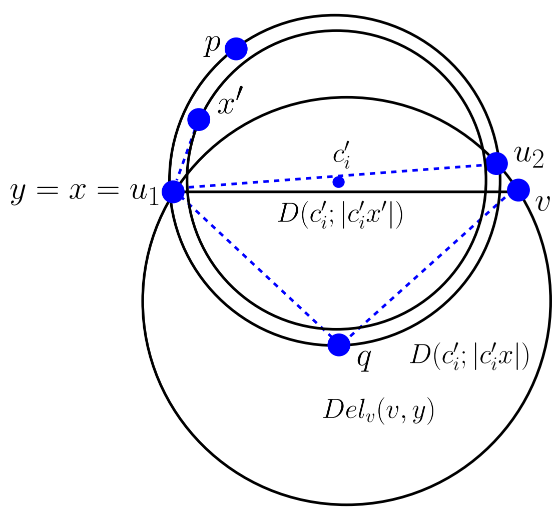
Let $\widehat{u_1 u_2}$ be the arc on $\partial D(x, p, q)$ with endpoints u_1 and u_2 that contains the north pole of $\partial D(x, p, q)$. Since $\angle u_1 q u_2 \leq \angle yqv < \pi/2$, $\widehat{u_1 u_2}$ is a minor arc.

Recall that $x \notin int(Del_v(v, y))$. Also, if $x \in \partial Del_v(v, y)$, then $x = y$. It follows that x is not below the line through u_1 and u_2. By a similar argument, x' is not below this line. Since both x and x' are in $D(x, p, q)$ and since $\widehat{u_1 u_2}$ is a minor arc, it follows from Lemma 3 that $|xx'| \leq 1$, which is a contradiction.

Thus, we have shown that $|c'_i x'| = |c'_i x|$. Recall that p is the point that is computed in line 13 of algorithm PLDG(v). Consider the point p' that is computed in line 14 of algorithm PLDG′(v). Since $D(c'_i; |c'_i x|) = D(c'_i; |c'_i x'|)$, we have $\{x, p\} = \{x', p'\}$. In other words, algorithm PLDG′(v) knows the points x and p, but does not know which of them is x and which of them is p.



(a) $y \neq x$



(b) $y = x$

Figure 6: *Illustrating the proof of Lemma 10.*

Since lines 15–19 of algorithm PLDG(v) are symmetric in x and p, and since lines 16–22 of algorithm PLDG$'(v)$ are symmetric in x' and p', it follows that the behaviors of PLDG(v) and PLDG$'(v)$ with respect to the edge (v, y) are identical. Therefore, algorithm PLDG$'(v)$ removes the edge (v, y) from $E'(v)$. ■

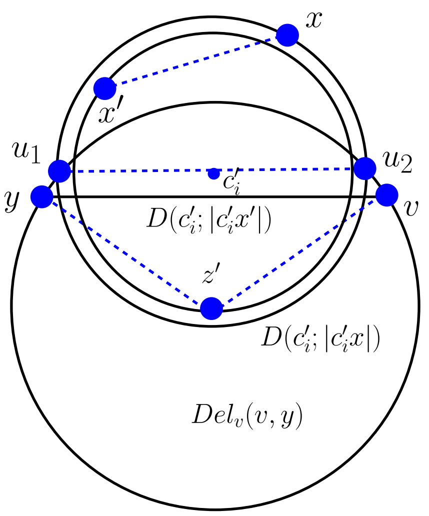
Lemma 11 *Let v be an element of V and let (v, y) be an edge of the Delaunay triangulation $LDT(v)$ of the set N_v. If algorithm* PLDG$'(v)$ *removes (v, y) from $E'(v)$, then algorithm* PLDG(v) *removes (v, y) from $E(v)$.*

Proof. Since algorithm PLDG$'(v)$ removes (v, y) from $E'(v)$, there exist three pairwise distinct points x, p, and q in V such that

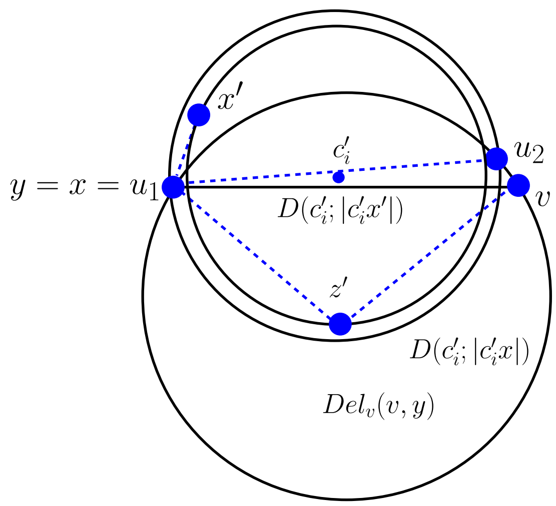
1. $\Delta(x, p, q)$ is a triangular face in $LDT(x)$,
2. algorithm PLDG$'(x)$ broadcasts the center c'_i of the disk $D(x, p, q) = D(c'_i; |c'_i x|)$,
3. $v \neq x$, $|vx| \leq 1$,
4. v receives the center c'_i (but does not know that it was broadcast by x).

Consider the point x' that is computed in line 11 of algorithm PLDG$'(v)$. Thus, x' is a point in $N_v \setminus \{v\}$ that is closest to c'_i. Since $x \in N_v \setminus \{v\}$, we have $|c'_i x'| \leq |c'_i x|$. In the rest of the proof, we will show that $|c'_i x'| = |c'_i x|$. As in the proof of Lemma 10, this will imply that algorithm PLDG(v) removes the edge (v, y) from $E(v)$.

The proof of the claim that $|c'_i x'| = |c'_i x|$ is by contradiction. Thus, we assume that $|c'_i x'| < |c'_i x|$. Then x' is in the interior of the disk $D(x, p, q)$. Since $\Delta(x, p, q)$ is a triangular face in $LDT(x)$, we have $|xx'| > 1$. Since $|vx| \leq 1$, v is not in the interior of $D(x, p, q)$.



(a) $y \neq x$



(b) $y = x$

Figure 7: *Illustrating the proof of Lemma 11.*

Consider the disk $D(c_i'; |c_i'x'|)$. It follows from line 13 of algorithm PLDG$'(v)$ that the boundary of this disk contains exactly two points of N_v; x' is one of these two points, let p' be the other one. Thus, p' is the point that is computed in line 14 of algorithm PLDG$'(v)$. Since $y \in N_v \setminus \{v\}$, the point y is not in the interior of $D(c_i'; |c_i'x'|)$.

Consider the disk $Del_v(v, y)$ that is computed in line 20 of algorithm PLDG$'(v)$. Then $N_v \cap \partial Del_v(v, y) = \{v, y\}$ and $N_v \cap int(Del_v(v, y)) = \emptyset$. Since $|vx'| \leq 1$ and $|vp'| \leq 1$, neither x' nor p' is in the interior of $Del_v(v, y)$.

Consider the point z' on $arc_i = (\partial D(c_i'; |c_i'x'|)) \setminus D(v; 1)$ that is computed in line 18 of algorithm PLDG$'(v)$. It follows from line 21 that z' is in the interior of $Del_v(v, y)$ and vy crosses at least one of $x'z'$ and $p'z'$.

Since lines 11–22 of algorithm PLDG$'(v)$ are symmetric with respect to x' and p', we may assume without loss of generality that vy crosses $x'z'$. Thus, $x' \notin \{v, y\}$ and x' and z' are on opposite sides of the line through v and y.

We may assume without loss of generality that vy is horizontal, v is to the right of y, x' is above the line through v and y, and z' is below this line.

We claim that y is in the interior of $D(x, p, q)$. The proof is by contradiction; thus, we assume that $y \notin int(D(x, p, q))$. Observe that $z' \in int(D(x, p, q))$ and recall that $z' \in int(Del_v(v, y))$. Since $|vz'| > 1$ and $|vy| \leq 1$, we have $\angle yz'v < \pi/2$. Therefore, the upper arc on $\partial Del_v(v, y)$ with endpoints y and v is a minor arc. Let u_1 and u_2 be the two intersection points between $\partial Del_v(v, y)$ and $\partial D(x, p, q)$; refer to Figure 7. It follows from Lemma 3 that $|u_1u_2| \leq 1$. Since $\angle u_1z'u_2 \leq \angle yz'v < \pi/2$, the upper arc on $\partial D(x, p, q)$ with endpoints u_1 and u_2 is a minor arc. Since both x and x' are in $D(x, p, q)$ and on or above the line through u_1 and u_2, it follows, again from Lemma 3, that $|xx'| \leq 1$, which is a contradiction.

Thus, we have shown that $y \in int(D(x, p, q))$. Since $\Delta(x, p, q)$ is a triangular face in

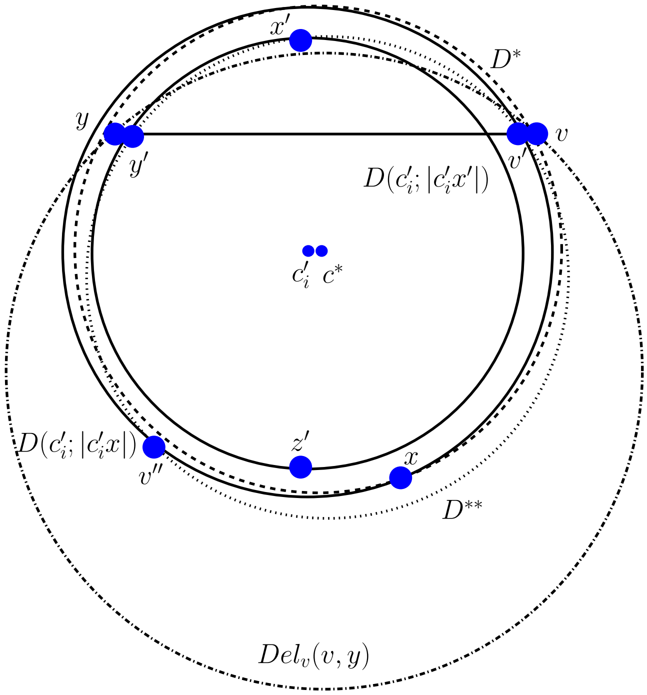


Figure 8: *An illustration of the proof of Lemma 11. The point y cannot be in the interior of $D(c_i'; |c_i'x|)$.*

$LDT(x)$, this implies that $|xy| > 1$.

We next claim that x is below the line through v and y. We prove this claim by contradiction. Thus, we assume that x is above this line. Observe that y is to the left of the vertical line through c_i'. We have seen above that $\angle yz'v < \pi/2$. Therefore, the arc on $\partial D(c_i'; |c_i'x'|)$ that is not below the line through v and y is a minor arc. It follows that c_i' is below the line through v and y.

Consider the disk $D(x, p, q)$. We translate the center c_i' of this disk horizontally to the right. During the translation, we change the disk so that x stays on its boundary. We stop the translation as soon as one of v and y is on the boundary of the moving disk. Let c^* be the center of the new disk D^*. Since c^* is below the line through v and y, the arc on ∂D^* that is not below the line through v and y is a minor arc. First assume that y is on the boundary of D^*. Then it follows from Lemma 3 that $|xy| \leq 1$, which is a contradiction. Thus, y is in the interior of D^* and v is on the boundary of D^*. Then, the disk $D(y; |yv|)$ contains the point x and, therefore, $|yx| \leq |yv| \leq 1$, which is also a contradiction.

We conclude that x is below the line through v and y. Let y' be the leftmost intersection between yv and $\partial D(c_i'; |c_i'x'|)$, and let v' be the intersection between yv and $\partial D(c_i'; |c_i'x|)$. We translate the center of the disk $D(c_i'; |c_i'x'|)$ along the line through y' and c_i', such that the center moves away from y'. During the translation, we change the disk so that y' stays on its boundary. We stop the translation as soon as v' is on the boundary of the moving disk; refer to Figure 8. Let D^{**} be the resulting disk. Let $v'' \neq v'$ be the second intersection point between $\partial D(c_i'; |c_i'x|)$ and ∂D^{**}. Then $|y'v'| = |y'v''|$. Since $|yv'| \leq |yv| \leq 1$ and $|yx| > 1$, the point x is not in the disk $D(y'; |y'v'|)$. Therefore, the point x is on the clockwise arc on $D(c_i'; |c_i'x|)$ from v' to v''. Observe that both x and x' are contained in D^{**}. It follows that any disk having y' and v' on its boundary contains at least one of x and x'. This, in turn, implies that any disk having y and v on its boundary contains at least one of x and x'. In particular, $Del_v(v, y)$ contains at least one of x and x', which is a contradiction. This completes the proof of the lemma. ■

By Lemmas 10 and 11, algorithms PLDG(v) and PLDG$'$(v) compute the same graph. Therefore, the proof of Theorem 1 can be completed as in the proof of Lemma 9 and by observing that the sequence that is broadcast in line 8 of algorithm PLDG$'$(v) contains at most 5 points.

4 Concluding Remarks

We have presented a 2-local algorithm that constructs the Plane Localized Delaunay Graph $PLDG(V)$ of any finite set V of points in the plane. This graph is a plane and consistent $\frac{4\pi\sqrt{3}}{9}$-spanner of the unit-disk graph $UDG(V)$. Our algorithm makes only one communication round and each point of V broadcasts at most 5 messages. We leave as an open problem the question of whether a 2-local algorithm exists in which each point broadcasts less than 5 messages.

In general, the maximum degree of any vertex in the graph $PLDG(V)$ can be linear in the size of V. It is still open whether there is a communication-efficient localized algorithm that constructs a bounded-degree plane spanner of the unit-disk graph.

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