π/2-ANGLE YAO GRAPHS ARE SPANNERS

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ABSTRACT

We show that the Yao graph Y_4 in the L_2 metric is a spanner with stretch factor $8\sqrt{2}(26+23\sqrt{2})$. Enroute to this, we also show that the Yao graph Y_4^∞ in the L_∞ metric is a plane spanner with stretch factor 8.

Keywords: Yao graph; Y4; spanner.

1. Introduction

Let V be a finite set of points in the plane and let $G = (V, E)$ be the complete Euclidean graph on V. We will refer to the points in V as *nodes*, to distinguish them from other points in the plane. The *Yao graph* [8] with an integer parameter $k > 0$, denoted Y_k, is defined as follows. Any k equally-separated rays starting at the origin define k cones. Pick a set of arbitrary, but fixed cones. Translate the cones to each node $u \in V$. In each cone with apex u, pick a shortest edge uv, if there is one, and add to Y_k the directed edge $\overrightarrow{uv}$. Ties are broken arbitrarily. Note that the Yao graph differs from the Θ-graph in how the shortest edge is chosen. While the Yao graph chooses the shortest edge in terms of the Euclidean distance, the Θ-graph chooses the edge whose projection on the bisector of the cone is shortest. Most of the time we ignore the direction of an edge uv; we refer to the directed version $\overrightarrow{uv}$ of uv only when its origin (u) is important and unclear from the context. We will distinguish between Y_k, the Yao graph in the Euclidean L_2 metric, and Y_k^∞, the Yao graph in the L_∞ metric. Unlike Y_k however, in constructing Y_k^∞ ties are broken by always selecting the *most clockwise* edge; the reason for this choice will become clear in Section 2.

The *length* of a path is the sum of the lengths of its constituent edges. For a given subgraph $H \subseteq G$ and a fixed $t \geq 1$, H is called a *t-spanner* for G if, for any two nodes $u, v \in V$, the shortest path in H from u to v is no longer than t times the length $|uv|$ of uv. The value t is called the *dilation* or the *stretch factor* of H. If t is constant, then H is called a *length spanner*, or simply a *spanner*.

The class of graphs Y_k has been much studied. Bose et al. [2] showed that, for $k \geq 9$, Y_k is a spanner with stretch factor $\frac{1}{\cos\frac{2\pi}{k} - \sin\frac{2\pi}{k}}$. In Ref. [1] we improved the stretch factor and showed that, in fact, Y_k is a spanner for any $k \geq 7$. Recently, Damian and Raudonis [4] showed that Y_6 is a 17.7-spanner. Molla [6] showed that Y_2 and Y_3 are not spanners, and that Y_4 is a spanner with stretch factor $4(2+\sqrt{2})$, for the special case when the nodes in V are in convex position (see also Ref. [3]). The authors conjectured that Y_4 is a spanner for arbitrary point sets. In this paper, we settle their conjecture and prove that Y_4 is a spanner with stretch factor $8\sqrt{2}(26+23\sqrt{2})$.

The paper is organized as follows. In Section 2, we prove that the graph Y_4^∞ is a spanner with stretch factor 8. In Section 3 we establish several properties for the graph Y_4. Finally, in Section 4, we use the properties of Section 3 to prove that, for every edge ab in Y_4^∞, there exists a path between a and b in Y_4 not much longer than the Euclidean distance between a and b. By combining this with the result of Section 2, we conclude that Y_4 is a spanner.

2. Y_4^∞ in the L_∞ Metric

In this section we focus on Y_4^∞, which has a nicer structure compared to Y_4. First we prove that Y_4^∞ is a plane graph. Then we use this property to show that Y_4^∞ is an 8-spanner. To be more precise, we prove that for any two nodes a and b, the graph Y_4^∞ contains a path between a and b whose length (in the L_∞-metric) is at most $8|ab|_\infty$.

We need a few definitions. We say that two edges ab and cd *properly cross* (or *cross*, for short) if they share a point other than an endpoint (a, b, c or d); we say that ab and cd *intersect* if they share a point (either an interior point or an endpoint).
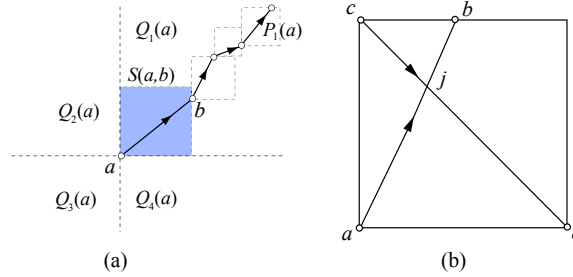

Fig. 1. (a) Definitions: $Q_i(a)$, $P_i(a)$ and $S(a,b)$. (b) Lemma 1: ab and cd cannot cross.

Throughout the paper, we will use the following notation: for each node $a \in V$, $x(a)$ is the x-coordinate of a and $y(a)$ is the y-coordinate of a; $Q_1(a)$, $Q_2(a)$, $Q_3(a)$ and $Q_4(a)$ are the four quadrants at a, depicted in Fig. 1a; each quadrant is half-open and half-closed, including all points on the clockwise boundary axis (with respect to the quadrant bisector through a), and excluding all points on the counterclockwise boundary axis; $P_i(a)$ is the path that starts at a and follows the directed Yao edges in quadrant Q_i; $P_i(a,b)$ is the subpath of $P_i(a)$ that starts at node a and ends at node b; $|ab|_\infty$ is the L_∞ distance between a and b, defined as $\max\{|x(a)-x(b)|, |y(a)-y(b)|\}$; $sp(a,b)$ is a shortest path in Y_4^∞ between a and b; $S(a,b)$ is the open square with corner a whose boundary contains b; and $\partial S(a,b)$ is the boundary of $S(a,b)$. These definitions are depicted in Fig. 1a.

Lemma 1. Y_4^∞ *is a plane graph.*

Proof. The proof is by contradiction. Assume the opposite. Then there are two edges $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4^\infty$ that cross each other. Since $\overrightarrow{ab} \in Y_4^\infty$, $S(a,b)$ must be empty of nodes in V, and similarly for $S(c,d)$. Let j be the intersection point between ab and cd. Then $j \in S(a,b) \cap S(c,d)$, meaning that $S(a,b)$ and $S(c,d)$ must overlap. However, neither square may contain a, b, c or d. It follows that $S(a,b)$ and $S(c,d)$ coincide, meaning that c and d lie on $\partial S(a,b)$ (see Fig. 1b). Since cd intersects ab,

c and d must lie on opposite sides of ab. Thus either ac or ad lies clockwise from ab. Assume without loss of generality that ad lies clockwise from ab; the other case is identical. Because $S(a, c)$ coincides with $S(a, b)$, we have that $|ad|_\infty = |ab|_\infty$. In this case however, Y_4^∞ would break the tie between ad and ab by selecting the most clockwise edge, which is $\overrightarrow{ad}$. (Note that, by the definition of a quadrant, ad belongs to the quadrant of a containing b.) This contradicts that $\overrightarrow{ab} \in Y_4^\infty$. □

Theorem 1. *Y_4^∞ is an 8-spanner in the L_∞ metric.*

Proof. We show that, for any pair of points $a, b \in V$, $|sp(a, b)|_\infty < 8|ab|_\infty$. The proof is by induction on the pairwise L_∞-distance between the points in V. Assume without loss of generality that $b \in Q_1(a)$, and $|ab|_\infty = |x(b) - x(a)|$ (i.e., b lies below the diagonal of $S(a, b)$ incident to a). Consider the case in which ab is a closest (in the L_∞ metric) pair of points in V. This is the base case for our induction. If $ab \in Y_4^\infty$, then $|sp(a, b)|_\infty = |ab|_\infty$. Otherwise, there must be $ac \in Y_4^\infty$, with $|ac|_\infty = |ab|_\infty$. Recall that Y_4^∞ breaks ties by always selecting the most clockwise edge, so ac must be clockwise of ab. Then c must lie on the right side of the right triangle formed by a, b, and the lower right corner of $S(a, b)$. If b lies strictly below the upper right corner of $S(a, b)$, or if c lies strictly above the lower right corner of $S(a, b)$, then $|bc|_\infty < |ab|_\infty$, contradicting our choice of ab. For similar reasons, no points in V may lie along the segment bc. This along with the fact that $b \in Q_2(c)$, $S(c, b)$ is empty, and cb is a most clockwise edge in $Q_2(c)$, implies that $cb \in Y_4$. Then (a, c, b) is a path in Y_4 from a to b no longer than $2|ab|_\infty$.
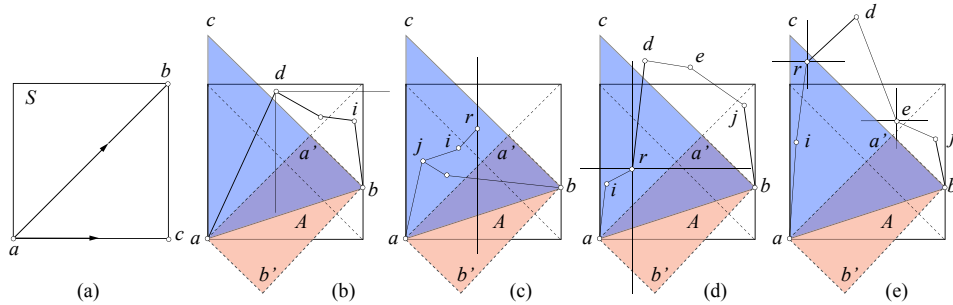

Fig. 2. (a) Base case. (b) $\triangle abc$ empty (c) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \{j\}$ (d) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \emptyset$, e above r (e) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \emptyset$, e below r.

Assume now that the inductive hypothesis holds for all pairs of points closer (in the L_∞ metric) than $|ab|_\infty$. If $ab \in Y_4^\infty$, then $|sp(a, b)|_\infty = |ab|_\infty$ and the proof is finished. If $ab \notin Y_4^\infty$, then the square $S(a, b)$ must be nonempty.

Let A be the rectangle $ab'ba'$ as in Fig. 2b, where ba' and bb' are parallel to the diagonals of $S(a, b)$. If A is nonempty, then we can use induction to prove

that $|sp(a,b)|_\infty \leq 8|ab|_\infty$ as follows. Pick $c \in A$ arbitrary. Then $|ac|_\infty + |cb|_\infty = |x(c)-x(a)|+|x(b)-x(c)| = |ab|_\infty$, and by the inductive hypothesis $sp(a,c)\oplus sp(c,b)$ is a path in Y_4^∞ no longer than $8|ac|_\infty + 8|cb|_\infty = 8|ab|_\infty$; here $\oplus$ represents the concatenation operator. Assume now that A is empty. Let c be at the intersection between the line supporting ba' and the vertical line through a (see Fig. 2b). We discuss two cases, depending on whether $\triangle abc$ is empty of points or not.

Case 1: $\triangle abc$ is empty of points. Let $ad \in P_1(a)$. We show that $P_4(d)$ cannot contain an edge crossing ab. Assume the opposite, and let $st \in P_4(d)$ cross ab. Note that $st \in P_4(d)$ also implies $st \in P_4(s)$, which along with the fact that st crosses ab, implies that s is either vertically aligned, or to the left of b.. Since $\triangle abc$ is empty, s must lie above bc and t below ab. It follows that b and t are in the same quadrant $Q_4(s)$ (recall that this quadrant includes the downward ray from s). Furthermore, $|st|_\infty \geq |y(s)-y(t)| > |y(s)-y(b)| = |sb|_\infty$, contradicting the fact that $st \in Y_4^\infty$.

We have established that $P_4(d)$ does not cross ab, which implies that $P_4(d)$ must exit $S(d,b)$ through its right edge. Also note that $P_2(b)$ cannot cross ac, because $\triangle abc$ is empty of points, and any point left of ac is L_∞-farther from b than d. It follows that $P_2(b)$ exits $S(b,d)$ through its top edge. This together with the fact that $P_4(d)$ exits $S(d,b)$ through its right edge, implies that $P_4(d)$ and $P_2(b)$ must meet in a point $i \in P_4(d) \cap P_2(b)$ (see Fig. 2b). Now note that $|P_4(d,i) \oplus P_2(b,i)|_\infty \leq |x(d)-x(b)|+|y(d)-y(b)| < 2|ab|_\infty$. Thus we have that $|sp(a,b)|_\infty \leq |ad \oplus P_4(d,i) \oplus P_2(b,i)|_\infty < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty$.

Case 2: $\triangle abc$ is nonempty. In this case, we seek a short path from a to b that does not cross to the underside of ab, to avoid oscillating paths that cross ab arbitrarily many times. Let r be the rightmost point that lies inside $\triangle abc$. Arguments similar to the ones used in Case 1 show that $P_3(r)$ cannot cross ab and therefore it must meet $P_1(a)$ in a point i. Then $P_{ar} = P_1(a,i) \oplus P_3(r,i)$ is a path in Y_4^∞ of length

$$|P_{ar}|_\infty < |x(a)-x(r)|+|y(a)-y(r)| < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty. \qquad (1)$$

The term $2|ab|_\infty$ in the inequality above results from the fact that $|y(a)-y(r)| \leq |y(a)-y(c)| \leq 2|ab|_\infty$. Consider first the simpler situation in which $P_2(b)$ meets P_{ar} in a point $j \in P_2(b) \cap P_{ar}$ (see Fig. 2c). Let $P_{ar}(a,j)$ be the subpath of P_{ar} extending between a and j. Then $P_{ar}(a,j) \oplus P_2(b,j)$ is a path in Y_4^∞ from a to b, therefore $|sp(a,b)|_\infty \leq |P_{ar}(a,j) \oplus P_2(b,j)|_\infty < 2|y(j)-y(a)| + |ab|_\infty \leq 5|ab|_\infty$.

Consider now the case when $P_2(b)$ does not intersect P_{ar}. We argue that, in this case, $Q_1(r)$ may not be empty. Assume the opposite. Then no edge $st \in P_2(b)$ may cross $Q_1(r)$. This is because, for any such edge, $|sr|_\infty < |st|_\infty$, contradicting $st \in Y_4^\infty$. This implies that $P_2(b)$ intersects P_{ar}, again a contradiction to our assumption. This establishes that $Q_1(r)$ is nonempty. Let $rd \in P_1(r)$. The fact that $P_2(b)$ does not intersect P_{ar} implies that d lies to the left of b. The fact that r is the rightmost point in $\triangle abc$ implies that d lies outside $\triangle abc$ (see Fig. 2d). It also implies that $P_4(d)$ shares no points with $\triangle abc$. This along with arguments similar to the ones

used in case 1 show that $P_4(d)$ and $P_2(b)$ meet in a point $j \in P_4(d) \cap P_2(b)$. Thus we have found a path

$$P_{ab} = P_1(a, i) \oplus P_3(r, i) \oplus rd \oplus P_4(d, j) \oplus P_2(b, j). \tag{2}$$

extending from a to b in Y_4^∞. If $|rd|_\infty = |x(d) - x(r)|$, then $|rd|_\infty < |x(b) - x(a)| = |ab|_\infty$, and the path P_{ab} has length

$$|P_{ab}|_\infty \le 2|y(d) - y(a)| + |ab|_\infty < 7|ab|_\infty. \tag{3}$$

In the above, we used the fact that $|y(d) - y(a)| = |y(d) - y(r)| + |y(r) - y(a)| < |ab|_\infty + 2|ab|_\infty$. Suppose now that

$$|rd|_\infty = |y(d) - y(r)|. \tag{4}$$

In this case, it is unclear whether the path P_{ab} defined by (2) is short, since rd can be arbitrarily long compared to ab. Let e be the clockwise neighbor of d along the path P_{ab} (e and b may coincide). Then e lies below d, and either $de \in P_4(d)$, or $ed \in P_2(e)$ (or both). If e lies above r, or at the same level as r (i.e., $e \in Q_1(r)$, as in Fig. 2d), then

$$|y(e) - y(r)| < |y(d) - y(r)|. \tag{5}$$

Since $rd \in P_1(r)$ and e is in the same quadrant of r as d, we have $|rd|_\infty \le |re|_\infty$. This along with inequalities (4) and (5) implies $|re|_\infty > |y(e) - y(r)|$, which in turn implies $|re|_\infty = |x(e) - x(r)| \le |ab|_\infty$, and so $|rd|_\infty \le |ab|_\infty$. Then inequality (3) applies here as well, showing that $|P_{ab}|_\infty < 7|ab|_\infty$.

If e lies below r (as in Fig. 2e), then

$$|ed|_\infty \ge |y(d) - y(e)| \ge |y(d) - y(r)| = |rd|_\infty. \tag{6}$$

Assume first that $ed \in P_2(e)$, or $|ed|_\infty = |x(e) - x(d)|$. In either case, $|ed|_\infty \le |er|_\infty < 2|ab|_\infty$. This along with inequality (6) shows that $|rd|_\infty < 2|ab|_\infty$. Substituting this upper bound in (2), we get $|P_{ab}|_\infty \le 2|y(d) - y(a)| + 2|ab|_\infty < 8|ab|_\infty$. Assume now that $ed \notin P_2(e)$, and $|ed|_\infty = |y(e) - y(d)|$. Then $ee' \in P_2(e)$ cannot go above d (otherwise $|ed|_\infty < |ee'|_\infty$, contradicting $ee' \in P_2(e)$). This along with the fact $de \in P_4(d)$ implies that $P_2(e)$ intersects P_{ar} in a point k. Redefine $P_{ab} = P_{ar}(a, k) \oplus P_2(e, k) \oplus P_4(e, j) \oplus P_2(b, j)$. Then P_{ab} is a path in Y_4^∞ from a to b of length $|P_{ab}| \le 2|y(r) - y(a)| + |ab|_\infty \le 5|ab|_\infty$. $\qquad \square$

This theorem will be employed in Section 4.

3. Y_4 in the L_2 Metric

In this section we establish basic properties of Y_4. The ultimate goal of this section is to show that, if two edges in Y_4 cross, there is a short path between their endpoints (Lemma 8). We begin with a few definitions.

Let $Q(a, b)$ denote the quadrant with origin at a that contains b. In other words, $Q(a, b) = Q_i(a)$, where $Q_i(a)$ contains b. For a pair of nodes $a, b \in V$, define

recursively a directed path $\mathcal{P}(a \to b)$ from a to b in Y_4 as follows. If $a = b$, then $\mathcal{P}(a \to b) = null$. If $a \neq b$, there must exist $\overrightarrow{ac} \in Y_4$ that lies in $Q(a,b)$. In this case, define

$$\mathcal{P}(a \to b) = \overrightarrow{ac} \oplus \mathcal{P}(c \to b).$$

Recall that $\oplus$ represents the concatenation operator. This definition is illustrated in Fig. 3a. Fischer et al. [5] show that $\mathcal{P}(a \to b)$ is well defined and lies entirely inside the square centered at b whose boundary contains a.
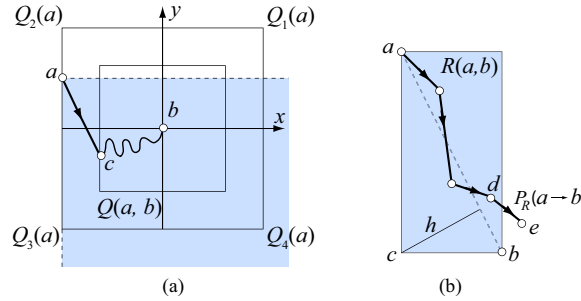

Fig. 3. Definitions. (a) $Q(a,b)$ and $\mathcal{P}(a \to b)$. (b) $\mathcal{P}_R(a \to b)$.

For any path P and any pair of nodes $a, b \in P$, let $P[a,b]$ be the subpath of P from a to b. Let $R(a,b)$ be the closed axis-aligned rectangle with diagonal ab "(we permit $R(a,b)$ to be degenerate rectangle, when ab is either horizontal or vertical).

For a fixed pair of nodes $a, b \in V$, define a path $\mathcal{P}_R(a \to b)$ as follows. Let $e \in V$ be the first node along $\mathcal{P}(a \to b)$ that is not strictly interior to $R(a,b)$. Then $\mathcal{P}_R(a \to b)$ is the subpath of $\mathcal{P}(a \to b)$ that extends between a and e. In other words, $\mathcal{P}_R(a \to b)$ is the path that follows the Y_4 edges pointing towards b, truncated as soon as it reaches b or leaves $R(a,b)$. Formally, $\mathcal{P}_R(a \to b) = \mathcal{P}(a \to b)[a,e]$. This definition is illustrated in Fig. 3b. Our proofs will make use of the following two propositions.

Proposition 1. *The sum of the lengths of crossing diagonals of a non-degenerate (necessarily convex) quadrilateral abcd is strictly greater than the sum of the lengths of either pair of opposite sides:*

$$|ac| + |bd| > |ab| + |cd|$$
$$|ac| + |bd| > |bc| + |da|.$$

This can be proved by partitioning the diagonals into two pieces each at their intersection point, and then applying the triangle inequality twice.

Proposition 2. *For any triangle $\triangle abc$, the following inequalities hold:*

$$|ac|^2 \begin{cases} < |ab|^2 + |bc|^2, & \text{if } \angle abc < \pi/2 \\ = |ab|^2 + |bc|^2, & \text{if } \angle abc = \pi/2 \\ > |ab|^2 + |bc|^2, & \text{if } \angle abc > \pi/2 \end{cases}$$

This proposition follows immediately from the Law of Cosines applied to triangle $\triangle abc$.

Lemma 2. *For each pair of nodes $a, b \in V$,*

$$|\mathcal{P}_R(a \to b)| \leq |ab|\sqrt{2}. \tag{7}$$

Furthermore, each edge of $\mathcal{P}_R(a \to b)$ is no longer than $|ab|$.

Proof. Let c be one of the two corners of $R(a, b)$, other than a and b. Let $\overrightarrow{de} \in \mathcal{P}_R(a \to b)$ be the last edge on $\mathcal{P}_R(a \to b)$, which necessarily intersects $\partial R(a, b)$ (note that it is possible that $e = b$). Refer to Fig. 3b. Then $|de| \leq |db|$, otherwise $\overrightarrow{de}$ could not be in Y_4. Since db lies in the rectangle with diagonal ab, we have that $|db| \leq |ab|$, and similarly for each edge on $\mathcal{P}_R(a \to b)$. This establishes the latter claim of the lemma. For the first claim of the lemma, let $p = \mathcal{P}_R(a \to b)[a, d] \oplus db$. Since $|de| \leq |db|$, we have that $|\mathcal{P}_R(a \to b)| \leq |p|$. Since p lies entirely inside $R(a, b)$ and consists of edges pointing towards b, we have that p is an xy-monotone path (i.e., any line parallel to a coordinate axis intersects p in at most one point). It follows that $|p| \leq |ac| + |cb|$, which is bounded above by $|ab|\sqrt{2}$. $\qquad\square$
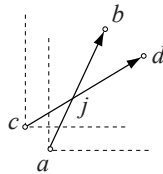

Fig. 4. Lemma 3: if ab and cd cross, they cannot both be in Y_4.

Lemma 3. *Let $a, b, c, d \in V$ be four disjoint nodes such that $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$, $b \in Q_i(a)$ and $d \in Q_i(c)$, for some $i \in \{1, 2, 3, 4\}$. Then ab and cd cannot cross.*

Proof. We may assume without loss of generality that $i = 1$ and c is to the left of a. The proof is by contradiction. Assume that ab and cd cross each other. Let j be the intersection point between ab and cd (see Fig. 4). Since $j \in Q_1(a) \cap Q_1(c)$,

it follows that $d \in Q_1(a)$ and $b \in Q_1(c)$. Thus $|ab| \le |ad|$, because otherwise, $\overrightarrow{ab}$ cannot be in Y_4. By Proposition 1 applied to the quadrilateral $adbc$,

$$|ad| + |cb| < |ab| + |cd|.$$

This along with $|ab| \le |ad|$ implies that $|cb| < |cd|$, contradicting that $\overrightarrow{cd} \in Y_4$. $\square$

The next four lemmas (4–7) each concern a pair of crossing Y_4 edges, culminating (in Lemma 8) in the conclusion that there is a short path in Y_4 between a pair of endpoints of those edges. We choose to defer the proofs of lemmas 4–6 to the appendix, for a better understanding of the logical flow of our analysis.

Lemma 4. *Let a, b, c and d be four disjoint nodes in V such that $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$, and ab crosses cd. Then (i) the ratio between the shortest side and the longer diagonal of the quadrilateral acbd is no greater than $1/\sqrt{2}$, and (ii) the shortest side of the quadrilateral acbd is strictly shorter than either diagonal.*

Lemma 5. *Let a, b, c, d be four distinct nodes in V, with $c \in Q_1(a)$, such that (i) $\overrightarrow{ab} \in Q_1(a)$ and $\overrightarrow{cd} \in Q_2(c)$ are in Y_4 and cross each other, and (ii) ad is a shortest side of quadrilateral acbd. Then $\mathcal{P}_R(a \to d)$ and $\mathcal{P}_R(d \to a)$ have a nonempty intersection.*

Lemma 6. *Let a, b, c, d be four distinct nodes in V, with $c \in Q_1(a)$, such that (i) $\overrightarrow{ab} \in Q_1(a)$ and $\overrightarrow{cd} \in Q_3(c)$ are in Y_4 and cross each other, and (ii) ad is a shortest side of quadrilateral acbd. Then $\mathcal{P}_R(d \to a)$ does not cross ab.*

The next lemma relies on all of Lemmas 2–6.

Lemma 7. *Let $a, b, c, d \in V$ be four distinct nodes such that $\overrightarrow{ab} \in Y_4$ crosses $\overrightarrow{cd} \in Y_4$, and let xy be a shortest side of the quadrilateral acbd. Then there exist two paths $\mathcal{P}_x$ and $\mathcal{P}_y$ in Y_4, where $\mathcal{P}_x$ has x as an endpoint and $\mathcal{P}_y$ has y as an endpoint, with the following properties:*

(i) $\mathcal{P}_x$ and $\mathcal{P}_y$ have a nonempty intersection.
(ii) $|\mathcal{P}_x| + |\mathcal{P}_y| \le 3\sqrt{2}|xy|$.
(iii) Each edge on $\mathcal{P}_x \cup \mathcal{P}_y$ is no longer than $|xy|$.

Proof. Assume without loss of generality that $b \in Q_1(a)$. We discuss the following exhaustive cases:

(1) $c \in Q_1(a)$, and $d \in Q_1(c)$. In this case, ab and cd cannot cross each other (by Lemma 3), so this case is finished.
(2) $c \in Q_1(a)$, and $d \in Q_2(c)$, as in Fig. 5a. Since $\overrightarrow{ab} \in Y_4$, $|ab| \le |ac|$. Since ab crosses cd, and $|ab| \le |ac|$, $b \in Q_2(c)$. Since $\overrightarrow{cd} \in Y_4$, $|cd| \le |cb|$. These along with Lemma 4 imply that ad and db are the only candidates for a shortest edge of $acbd$. Assume first that ad is a shortest edge of $acbd$. By Lemma 3,
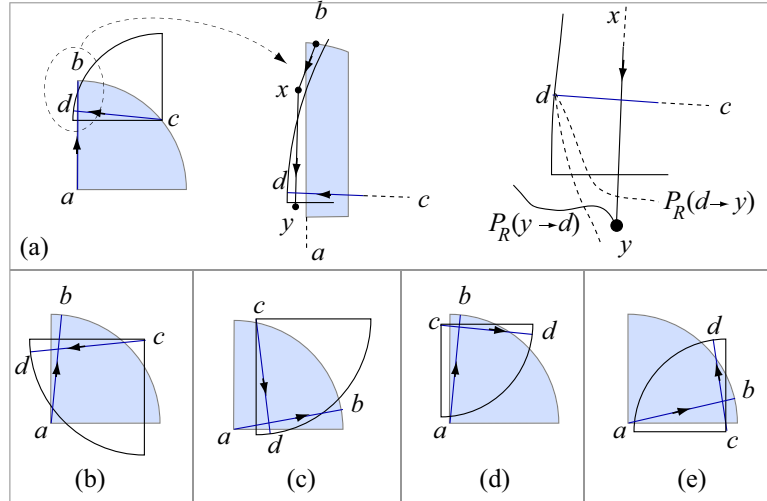
Fig. 5. Lemma 7: (a, b) $c \in Q_1(a)$ (c) $c \in Q_2(a)$ (d) $c \in Q_4(a)$.

$\mathcal{P}_a = \mathcal{P}_R(a \to d)$ does not cross cd, because $\mathcal{P}_a \in Q_2(a)$ and $cd \in Q_2(c)$ are in the quadrants of identical indices. It follows from Lemma 5 that $\mathcal{P}_a$ and $\mathcal{P}_d = \mathcal{P}_R(d \to a)$ have a nonempty intersection. Furthermore, by Lemma 2, $|\mathcal{P}_a| \leq |ad|\sqrt{2}$ and $|\mathcal{P}_d| \leq |ad|\sqrt{2}$, and no edge on these paths is longer than $|ad|$, proving the lemma true for this case. Consider now the case when db is a shortest edge of $acbd$ (see Fig. 5a). Note that d is below b (otherwise, $d \in Q_2(c)$ and $|cd| > |cb|$) and, therefore, $b \in Q_1(d)$. By Lemma 3, $\mathcal{P}_d = \mathcal{P}_R(d \to b)$ does not cross ab, because $\mathcal{P}_d \in Q_1(d)$ and $ab \in Q_1(a)$. If $\mathcal{P}_b = \mathcal{P}_R(b \to d)$ does not cross cd, then $\mathcal{P}_b$ and $\mathcal{P}_d$ have a nonempty intersection, proving the lemma true for this case. Otherwise, there exists $\overrightarrow{xy} \in \mathcal{P}_R(b \to d)$ that crosses cd (see Fig. 5a). Define

$$\mathcal{P}_b = \mathcal{P}_R(b \to d) \oplus \mathcal{P}_R(y \to d)$$
$$\mathcal{P}_d = \mathcal{P}_R(d \to y).$$

By Lemma 3, $\mathcal{P}_R(y \to d)$ does not cross cd, because they are both in quadrant Q_2. Then $\mathcal{P}_b$ and $\mathcal{P}_d$ must have a nonempty intersection. We now show that $\mathcal{P}_b$ and $\mathcal{P}_d$ satisfy conditions (i) and (iii) of the lemma. Proposition 1 applied on the quadrilateral $xdyc$ tells us that $|xc| + |yd| < |xy| + |cd|$. We also have that $|cx| \geq |cd|$, since $\overrightarrow{cd} \in Y_4$ and x is in the same quadrant of c as d. This along with the inequality above implies $|yd| < |xy|$. Because $xy \in \mathcal{P}_R(b \to d)$, by Lemma 2 we have that $|xy| \leq |bd|$, which along with the previous inequality shows that $|yd| < |bd|$. This along with Lemma 2 shows that condition (iii) of the lemma is satisfied. Furthermore, $|\mathcal{P}_R(y \to d)| \leq |yd|\sqrt{2}$ and $|\mathcal{P}_R(d \to y)| \leq |yd|\sqrt{2}$. It follows that $|\mathcal{P}_b| + |\mathcal{P}_d| \leq 3\sqrt{2}|bd|$.

(3) $c \in Q_1(a)$, and $d \in Q_3(c)$, as in Fig. 5b. Then $|ac| \geq \max\{ab, cd\}$, and by Lemma 4 ac is not a shortest edge of $acbd$. The case when bd is a shortest edge of $acbd$ is settled by Lemmas 3 and 2: Lemma 3 tells us that $\mathcal{P}_d = \mathcal{P}_R(d \to b)$ does not cross ab, (because they are both in Q_1,) and $\mathcal{P}_b = \mathcal{P}_R(b \to d)$ does not cross cd (because they are both in Q_3). It follows that $\mathcal{P}_d$ and $\mathcal{P}_b$ have a nonempty intersection. Furthermore, Lemma 2 guarantees that $\mathcal{P}_d$ and $\mathcal{P}_b$ satisfy conditions (ii) and (iii) of the lemma. Consider now the case when ad is a shortest edge of $acbd$; the case when bc is shortest is symmetric. By Lemma 6, $\mathcal{P}_R(d \to a)$ does not cross ab. If $\mathcal{P}_R(a \to d)$ does not cross cd, then this case is settled: $\mathcal{P}_d = \mathcal{P}_R(d \to a)$ and $\mathcal{P}_a = \mathcal{P}_R(a \to d)$ satisfy the three conditions of the lemma. Otherwise, let $\overrightarrow{xy} \in \mathcal{P}_R(a \to d)$ be the edge crossing cd. Arguments similar to the ones used in case 1 above show that $\mathcal{P}_a = \mathcal{P}_R(a \to d) \oplus \mathcal{P}_R(y \to d)$ and $\mathcal{P}_d = \mathcal{P}_R(d \to y)$ are two paths that satisfy the conditions of the lemma.

(4) $c \in Q_1(a)$, and $d \in Q_4(c)$, as in Fig. 5c. Note that a horizontal reflection of Fig. 5c, followed by a rotation of $\pi/2$, depicts a case identical to case (2), Fig. 5a, which has already been settled.

(5) $c \in Q_2(a)$, as in Fig. 5d. Note that Fig. 5d rotated by $\pi/2$ depicts a case identical to case (2), Fig. 5a (with the roles of ab and cd switched), which has already been settled.

(6) $c \in Q_3(a)$. Then it must be that $d \in Q_1(c)$, otherwise cd cannot cross ab. By Lemma 3 however, ab and cd may not cross, unless one of them is not in Y_4.

(7) $c \in Q_4(a)$. By Lemma 3, d may not lie in $Q_1(c)$, therefore d must be in $Q_2(c)$, as in Fig. 5e. Note that a vertical reflection of Fig. 5e depicts a case identical to case (2), Fig. 5a (with the roles of ab and cd switched), so this case is settled as well. $\qquad\square$

We are now ready to establish the main lemma of this section, showing that there is a short path between the endpoints of two intersecting edges in Y_4.

Lemma 8. *Let* $a, b, c, d \in V$ *be four distinct nodes such that* $\overrightarrow{ab} \in Y_4$ *crosses* $\overrightarrow{cd} \in Y_4$*, and let* xy *be a shortest side of the quadrilateral* $acbd$*. Then* Y_4 *contains a path* $p(x, y)$ *connecting* x *and* y*, of length* $|p(x, y)| \leq \frac{6}{\sqrt{2}-1} \cdot |xy|$*. Furthermore, no edge on* $p(x, y)$ *is longer than* $|xy|$*.*

Proof. Let $\mathcal{P}_x$ and $\mathcal{P}_y$ be the two paths whose existence in Y_4 is guaranteed by Lemma 7. By condition (iii) of Lemma 7, no edge on $\mathcal{P}_x$ and $\mathcal{P}_y$ is longer than $|xy|$. By condition (i) of Lemma 7, $\mathcal{P}_x$ and $\mathcal{P}_y$ have a nonempty intersection. If $\mathcal{P}_x$ and $\mathcal{P}_y$ share a node $u \in V$, then the path $p(x, y) = \mathcal{P}_x[x, u] \oplus \mathcal{P}_y[y, u]$ is a path from x to y in Y_4 no longer than $3\sqrt{2}|xy|$; the length restriction follows from guarantee (ii) of Lemma 7. Otherwise, let $\overrightarrow{a'b'} \in \mathcal{P}_x$ and $\overrightarrow{c'd'} \in \mathcal{P}_y$ be two edges crossing each other. Let $x'y'$ be a shortest side of the quadrilateral $a'c'b'd'$, with $x' \in \mathcal{P}_x$ and $y' \in \mathcal{P}_y$. Lemma 7 tells us that $|a'b'| \leq |xy|$ and $|c'd'| \leq |xy|$. These along with

Lemma 4 imply that

$$|x'y'| \leq |xy|/\sqrt{2}. \tag{8}$$

This enables us to derive a recursive formula for computing a path $p(x, y) \in Y_4$ as follows:

$$p(x, y) = \begin{cases} x, & \text{if } x = y \\ \mathcal{P}_x[x, x'] \oplus \mathcal{P}_y[y, y'] \oplus p(x', y'), & \text{if } x \neq y. \end{cases} \tag{9}$$

Next we use induction on the length of xy to prove the claim of the lemma. The base case corresponds to $x = y$. In this case $p(x, y)$ degenerates to a point and $|p(x, y)| = 0$. To prove the inductive step, pick a shortest side xy of a quadrilateral $acbd$, with $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$ crossing each other, and assume that the lemma holds for all such sides shorter than xy. Let $p(x, y)$ be the path determined recursively as in (9). By the inductive hypothesis, we have that $p(x', y')$ contains no edges longer than $|x'y'| \leq |xy|$, and

$$|p(x', y')| \leq \frac{6}{\sqrt{2} - 1} |x'y'| \leq \frac{6}{2 - \sqrt{2}} |xy|. \tag{10}$$

This latter inequality follows from (8). Also recall that no edge on $\mathcal{P}_x$ and $\mathcal{P}_y$ is longer than $|xy|$, which together with formula (9) and the arguments above, implies that no edge on $p(x, y)$ is longer than $|xy|$. Substituting inequalities 10 and (ii) from Lemma 7 in formula (9) yields

$$|p(x, y)| \leq (3\sqrt{2} + \frac{6}{2 - \sqrt{2}}) \cdot |xy| = \frac{6}{\sqrt{2} - 1} \cdot |xy|.$$

This completes the proof.   □

4. Y_4^∞ and Y_4

The final step of our analysis is to prove that every individual edge of Y_4^∞ is spanned by a short path in Y_4. This, along with the result of Theorem 1, establishes that Y_4 is a spanner.

Fix an edge $\overrightarrow{ab} \in Y_4^\infty$. Call an edge or a path *t-short* (with respect to $|ab|$) if its length is within a constant factor t of $|ab|$. In our proof that ab is spanned by a t-short path in Y_4, we will make use of the following three statements (proved in the Appendix).

S1 If xy is t-short, then $\mathcal{P}_R(x \rightarrow y)$, and therefore its reverse, $\mathcal{P}_R^{-1}(x \rightarrow y)$ are $t\sqrt{2}$-short by Lemma 2.

S2 If $xy \in Y_4$ is t_1-short and $zw \in Y_4$ is t_2-short, and if xy intersects zw, Lemma 4(ii) and Lemma 8 show that there is a t_3-short path between any two of the endpoints of these edges, with $t_3 = t_1 + t_2 + 3(2 + \sqrt{2}) \max(t_1, t_2)$.

S3 If $p(x, y)$ is a t_1-short path and $p(z, w)$ is a t_2-short path and these two paths intersect, then by **S2** there is a t_3-short path P between any two of the endpoints of these paths, with $t_3 = t_1 + t_2 + 3(2 + \sqrt{2}) \max(t_1, t_2)$.

Lemma 9. *Fix an edge $ab \in Y_4^\infty$. There is a path $p(a,b) \in Y_4$ between a and b, of length $|p(a,b)| \leq t|ab|$, for $t = 26 + 23\sqrt{2}$.*

Proof. For the sake of clarity, we only prove here that there is a short path $p(a, b)$ between a and b, and and defer the calculation of the actual stretch factor t to the Appendix. We refer to an edge or a path as *short* if its length is within a constant factor of $|ab|$. Assume without loss of generality that $\overrightarrow{ab} \in Q_1(a)$. If $\overrightarrow{ab} \in Y_4$, then $p(a,b) = ab$ and the proof is finished. So assume the opposite, and let $\overrightarrow{ac}$ be the edge in Y_4 that lies in $Q_1(a)$; since $Q_1(a)$ is nonempty, $\overrightarrow{ac}$ exists. Because $\overrightarrow{ac} \in Y_4$ and b is in the same quadrant of a as c, we have that

$$
\begin{aligned}
|ac| &\leq |ab| &&\text{(i)} \\
|bc| &< |ac|\sqrt{2} &&\text{(ii).}
\end{aligned}
\tag{11}
$$

Inequality (ii) above follows immediately from the Law of Cosines, which implies that $|bc|^2 < |ab|^2 + |ac|^2$ (because the angle formed by ab and ac is strictly less than $\pi/2$), and the fact that $|ac| \leq |ab|$. Thus both ac and bc are short. And this in turn implies that $\mathcal{P}_R(b \to c)$ is short by **S1**. We next focus on $\mathcal{P}_R(b \to c)$. For simplicity, we assume that ac is counterclockwise of ab; the situation when ac lies clockwise of ab is symmetrical. Let $b' \notin R(b,c)$ be the other endpoint of $\mathcal{P}_R(b \to c)$. We distinguish three cases.
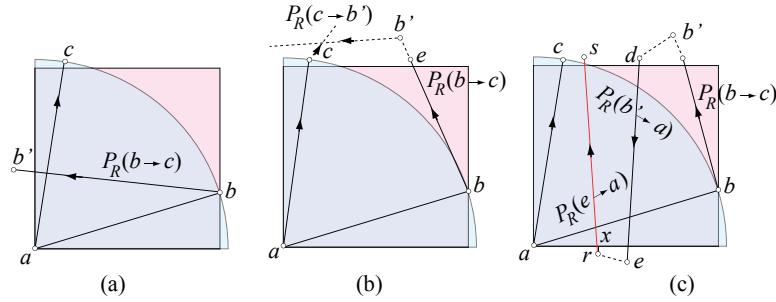

Fig. 6. Lemma 9: (a) Case 1: $\mathcal{P}_R(b \to c)$ and ac have a nonempty intersection. (b) Case 2: $\mathcal{P}_R(b' \to a)$ and ab have an empty intersection. (c) Case 3: $\mathcal{P}_R(b' \to a)$ and ab have a non-empty intersection.

Case 1: $\mathcal{P}_R(b \to c)$ and ac intersect (see Fig. 6a). Then by **S3** there is a short path $p(a,b)$ between a and b.

Case 2: $\mathcal{P}_R(b \to c)$ and ac do not intersect, and $\mathcal{P}_R(b' \to a)$ and ab do not intersect (see Fig. 6b). Note that because b' is the endpoint of the short path $\mathcal{P}_R(b \to c)$, the triangle inequality on $\triangle abb'$ implies that ab' is short, and therefore $\mathcal{P}_R(b' \to a)$ is short, by **S1**. We consider two cases:

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(i) $\mathcal{P}_R(b' \to a)$ intersects ac. Then by **S3** there is a short path $p(a, b')$. So

$$p(a, b) = p(a, b') \oplus \mathcal{P}_R^{-1}(b \to c)$$

   is short.

(ii) $\mathcal{P}_R(b' \to a)$ does not intersect ac. Then $\mathcal{P}_R(c \to b')$ must intersect $\mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$. Next we establish that $b'c$ is short. Let $\overrightarrow{eb'}$ be the last edge of $\mathcal{P}_R(b \to c)$, and so incident to b' (note that e and b may coincide). Because $\mathcal{P}_R(b \to c)$ does not intersect ac, b' and c are in the same quadrant for e. It follows that $|eb'| \leq |ec|$ and $\angle b'ec < \pi/2$. These observations along with Proposition 2 for $\triangle b'ec$ imply that $|b'c|^2 < |b'e|^2 + |ec|^2 \leq 2|ec|^2 < 2|bc|^2$ (this latter inequality uses the fact that $\angle bec > \pi/2$, which implies that $|ec| < |bc|$). It follows that

$$|b'c| \leq |bc|\sqrt{2} \ \leq \ 2|ac| \qquad \text{(by (11)ii).} \qquad (12)$$

   Thus $b'c$ is short, and by **S1** we have that $\mathcal{P}_R(c \to b')$ is short. Since $\mathcal{P}_R(c \to b')$ intersects the short path $\mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$, there is by **S3** a short path $p(c, b)$, and so

$$p(a, b) = ac \oplus p(c, b)$$

   is short.

Case 3: $\mathcal{P}_R(b \to c)$ and ac do not intersect, and $\mathcal{P}_R(b' \to a)$ intersects ab (see Fig. 6c). If $\mathcal{P}_R(b' \to a)$ intersects ab at a, then $p(a, b) = \mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$ is short. So assume otherwise, in which case there is an edge $\overrightarrow{de} \in \mathcal{P}_R(b' \to a)$ that crosses ab. Then $d \in Q_1(a)$, $e \in Q_3(a) \cup Q_4(a)$, and e and a are in the same quadrant for d. Note however that e cannot lie in $Q_3(a)$, since in that case $\angle dae > \pi/2$, which would imply $|de| > |da|$, which in turn would imply $\overrightarrow{de} \notin Y_4$. So it must be that $e \in Q_4(a)$.

   Next we show that $\mathcal{P}_R(e \to a)$ does not cross ab. Assume the opposite, and let $\overrightarrow{rs} \in \mathcal{P}_R(e \to a)$ cross ab. Then $r \in Q_4(a)$, $s \in Q_1(a) \cup Q_2(a)$, and s and a are in the same quadrant for r. Arguments similar to the ones above show that $s \notin Q_2(a)$, so s must lie in $Q_1(a)$. Let δ be the L_∞ distance from a to b. Let x be the projection of r on the horizontal line through a. Then

$$|rs| \geq |rx| + \delta \geq |rx| + |xa| > |ra| \quad \text{(by the triangle inequality)}$$

Because a and s are in the same quadrant for r, the inequality above contradicts $\overrightarrow{rs} \in Y_4$.

   We have established that $\mathcal{P}_R(e \to a)$ does not cross ab. Then $\mathcal{P}_R(a \to e)$ must intersect $\mathcal{P}' = de \oplus \mathcal{P}_R(e \to a)$. Note that de is short because it is in the short path $\mathcal{P}_R(b' \to a)$. Thus ae is short (because $|ae| < |ai| + |ei| < |ab| + |ed|$, where i is the intersection point between ab and de), and so $\mathcal{P}_R(a \to e)$ and $\mathcal{P}_R(e \to a)$ are short,

by **S1**. Then the short path $\mathcal{P}_R(a \to e)$ intersects either de or $\mathcal{P}_R(e \to a)$, each of which is short, and by **S3** there is a short path $p(a, e)$. Then

$$p(a, b) = p(a, e) \oplus \mathcal{P}_R^{-1}(b' \to a) \oplus \mathcal{P}_R^{-1}(b \to c)$$

is short. Straightforward calculations detailed in the appendix show that, in each of these cases, the stretch factor for $p(a, b)$ does not exceed $26 + 23\sqrt{2}$. $\square$

Our main result follows immediately from Theorem 1 and Lemma 9:

Theorem 2. Y_4 *is a t-spanner, for* $t \geq 8\sqrt{2}(26 + 23\sqrt{2})$.

5. Conclusion

Our results settle a long-standing open problem, asking whether Y_4 is a spanner or not. We answer this question positively, and establish a loose stretch factor of $8\sqrt{2}(26 + 23\sqrt{2})$. Finding tighter stretch factors for both Y_4^∞ and Y_4 remain interesting open problems. Establishing whether or not Y_5 is a spanner is also open.

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6. Appendix

6.1. *Proof of Lemma 4*

For any node $a \in V$, let $D(a, r)$ denote the open disk centered at a of radius r, and let $\partial D(a, r)$ denote the boundary of $D(a, r)$.

Proof. The first part of the lemma is a well-known fact that holds for any quadrilateral (see Ref. [7], for instance). For the second part of the lemma, let ab be the shorter of the diagonals of $acbd$, and assume without loss of generality that $\overrightarrow{ab} \in Q_1(a)$. Imagine two disks $D_a = D(a, |ab|)$ and $D_b = D(b, |ab|)$, as in Fig. 7a. If either c or d belongs to $D_a \cup D_b$, then the lemma follows: a shortest quadrilateral edge is shorter than $|ab|$.
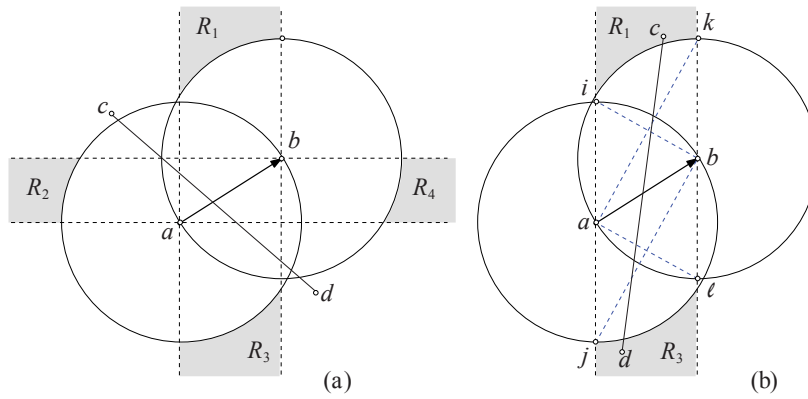

Fig. 7. Lemma 4 (a) $c \notin R_1 \cup R_2 \cup R_3 \cup R_4$ (b) $c \in R_1$.

So suppose that neither c nor d lies in $D_a \cup D_b$. In this case, we use the fact that cd crosses ab to show that $\overrightarrow{cd}$ cannot be an edge in Y_4. Define the following regions (see Fig. 7a):

$$R_1 = (Q_1(a) \cap Q_2(b)) \backslash (D_a \cup D_b)$$
$$R_2 = (Q_2(a) \cap Q_3(b)) \backslash (D_a \cup D_b)$$
$$R_3 = (Q_4(a) \cap Q_3(b)) \backslash (D_a \cup D_b)$$
$$R_4 = (Q_1(a) \cap Q_4(b)) \backslash (D_a \cup D_b).$$

If the node c is not inside any of the regions R_i, for $i = \{1, 2, 3, 4\}$, then the nodes a and b are in the same quadrant of c as d. In this case, note that either $\angle cad > \pi/2$ or $\angle cbd > \pi/2$, which implies that either $|ca|$ or $|cb|$ is strictly smaller than $|cd|$. These together show that $\overrightarrow{cd} \notin Y_4$.

So assume that c is in R_i for some $i \in \{1, 2, 3, 4\}$. In this situation, the node d must lie in the region R_j, with $j = (i + 2) \mod 4$ (with the understanding that

$R_0 = R_4$), because otherwise, either (i) a and d are in the same quadrant of c and $|ca| < |cd|$ or (ii) b and d are in the same quadrant of c and $|cb| < |cd|$. Either case contradicts the fact $\overrightarrow{cd} \in Y_4$. Consider now the case $c \in R_1$ and $d \in R_3$; the other cases are treated similarly. Let i and j be the intersection points between D_a and the vertical line through a. Similarly, let k and ℓ be the intersection points between D_b and the vertical line through b (see Fig. 7b). Since ij is a diameter of D_a, we have that $\angle ibj = \pi/2$ and similarly $\angle kal = \pi/2$. Also note that $\angle cbd \geq \angle ibj = \pi/2$, meaning that $|cd| > |cb|$. Similarly, $\angle cad \geq \angle kal = \pi/2$, meaning that $|cd| > |ca|$. These along with the fact that at least one of a and b is in the same quadrant for c as d, imply that $\overrightarrow{cd} \notin Y_4$. This completes the proof. $\square$

6.2. *Proof of Lemma 5*

Proof. The proof consists of two parts showing that the following claims hold: (I) $d \in Q_2(a)$ and (II) $\mathcal{P}_R(d \to a)$ does not cross ab. Before we prove these two claims, let us argue that they are sufficient to prove the lemma. Lemma 3 and claim (I) imply that $\mathcal{P}_R(a \to d)$ cannot cross cd, because $\mathcal{P}_R(a \to d) \in Q_2(a)$ and $cd \in Q_2(c)$ are in quadrants of identical indices. As a result, $\mathcal{P}_R(a \to d)$ intersects the left side of the rectangle $R(d, a)$. Consider the last edge $\overrightarrow{xy}$ of the path $\mathcal{P}_R(d \to a)$. If this edge crosses the right side of $R(a, d)$, then claim (II) implies that y is in the wedge bounded by ab and the upwards vertical ray starting at a; this further implies that $|ay| < |ab|$, contradicting the fact that $\overrightarrow{ab}$ is an edge in Y_4. Therefore, $\overrightarrow{xy}$ intersects the bottom side of $R(d, a)$, and the lemma follows (see Fig. 8b).

To prove the first claim (I), we observe that the lemma assumptions imply that $d \in Q_1(a) \cup Q_2(a)$. Therefore, it suffices to prove that d is not in $Q_1(a)$. Assume to the contrary that $d \in Q_1(a)$. Since $c \in Q_1(a)$, it must be that $b \in Q_2(c)$; otherwise, $\angle acb \geq \pi/2$, which implies $|ab| > |ac|$, contradicting the fact that $\overrightarrow{ab} \in Y_4$. Let i and j be the intersection points between cd and $\partial D(a, |ab|)$, where i is to the left of j. Since $\angle dbc \geq \angle ibj > \pi/2$, we have $|cb| < |cd|$. This, together with the fact that b and d are in the same quadrant $Q_2(c)$, contradicts the assumption that $\overrightarrow{cd}$ is an edge in Y_4. This completes the proof of claim (I).

Next we prove claim (II) by contradiction. Thus, we assume that there is an edge $\overrightarrow{xy}$ on the path $\mathcal{P}_R(d \to a)$ that crosses ab. Then necessarily $x \in R(a, d)$ and $y \in Q_1(a) \cup Q_4(a)$. If $y \in Q_4(a)$, then $\angle xay > \pi/2$, meaning that $|xy| > |xa|$, a contradiction to the fact that $\overrightarrow{xy} \in Y_4$. Thus, it must be that $y \in Q_1(a)$, as in Fig. 8a. This implies that $|ab| \leq |ay|$, because $\overrightarrow{ab} \in Y_4$.

The contradiction to our assumption that $\overrightarrow{xy}$ crosses ab will be obtained by proving that $|xy| > |xa|$. Indeed, this inequality contradicts the fact that $\overrightarrow{xy} \in Y_4$, because both a and y are in $Q_4(x)$, and Y_4 would have picked $\overrightarrow{xa}$ in place of $\overrightarrow{xy}$.

Let δ be the distance from x to the horizontal line through a. Our intermediate goal is to show that

$$\delta \leq |ab|/\sqrt{2}. \tag{13}$$

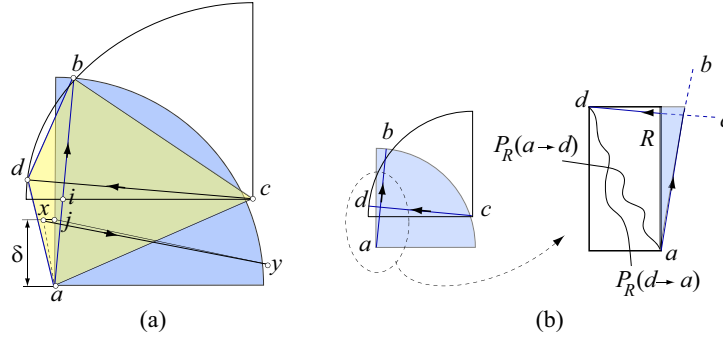18   *Bose, Damian, Douïeb, O'Rourke, Seamone, Smid and Wuhrer*

Fig. 8. (a) Lemma 5: $xy \in \mathcal{P}_R(d \to a)$ cannot cross ab.

We claim that $\angle acb < \pi/2$. Indeed, if this is not the case, then $|ac| < |ab|$, contradicting the fact that $\overrightarrow{ab}$ is an edge in Y_4. By a similar argument, and using the fact that $\overrightarrow{cd}$ is an edge in Y_4, we obtain the inequality $\angle cbd < \pi/2$. We now consider two cases, depending on the relative lengths of ac and cb.

(1) Assume first that $|ac| > |cb|$. If $\angle cad \geq \pi/2$, then $|cd| \geq |ac| > |cb|$, contradicting the fact that $\overrightarrow{cd}$ is an edge in Y_4 (recall that b and d are in the same quadrant of c). Therefore, we have $\angle cad < \pi/2$. So far we have established that three angles of the convex quadrilateral $acbd$ are acute. It follows that the fourth one ($\angle adb$) is obtuse. Proposition 2 applied to $\triangle adb$ tells us that

$$|ab|^2 > |ad|^2 + |db|^2 \geq 2|ad|^2,$$

where the latter inequality follows from the assumption that ad is a shortest side of $acbd$ (and, therefore, $|db| \geq |ad|$). Thus, we have that $|ad| \leq |ab|/\sqrt{2}$. This along with the fact that $x \in R(a, d)$ implies inequality (13).

(2) Assume now that $|ac| \leq |cb|$. Let i be the intersection point between ab and the horizontal line through c (refer to Fig. 8a). Note that $\angle aic \geq \pi/2$ and $\angle bic \leq \pi/2$ (these two angles sum to π). This along with Proposition 2 applied to triangle $\triangle aic$ shows that

$$|ac|^2 \geq |ai|^2 + |ic|^2.$$

Similarly, Proposition 2 applied to triangle $\triangle bic$ shows that

$$|bc|^2 \leq |bi|^2 + |ic|^2.$$

The two inequalities above along with our assumption that $|ac| \leq |cb|$ imply that $|ai| \leq |bi|$, which in turn implies that $|ai| \leq |ab|/2$, because $|ai|+|ib| = |ab|$. Since x is below i (otherwise, $|cx| < |cd|$, contradicting the fact that $\overrightarrow{cd}$ is an edge in Y_4), we have $\delta \leq |ai|$. It follows that $\delta \leq |ab|/2$.

Finally we derive a contradiction using the now established inequality (13). Let j be the orthogonal projection of x onto the vertical line through a (thus $|aj| = \delta$).

Note that $\angle ajy < \pi/2$, because $y \in Q_4(x)$. By Proposition 2 applied to $\triangle ajy$, we have

$$|ay|^2 < |aj|^2 + |jy|^2 = \delta^2 + |jy|^2.$$

Since y and b are in the same quadrant of a, and since $\overrightarrow{ab} \in Y_4$, we have that $|ab| \leq |ay|$. This along with the inequality above and (13) implies that $|jy| \geq |ab|/\sqrt{2} \geq \delta$. By Proposition 2 applied to $\triangle xjy$, we have $|xy|^2 > |xj|^2 + |jy|^2 \geq |xj|^2 + \delta^2 = |xj|^2 + |ja|^2 = |xa|^2$. It follows that $|xy| > |xa|$, contradicting our assumption that $\overrightarrow{xy} \in Y_4$. $\qquad\square$

6.3. *Proof of Lemma 6*

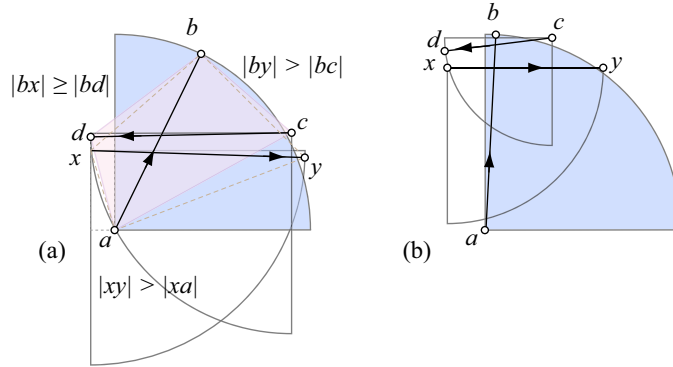**Proof.** We first show that $d \notin Q_3(a)$. Assume the opposite. Since $c \in Q_1(a)$ and

Fig. 9. Lemma 6: (a) $\mathcal{P}_R(d \to a)$ does not cross ab. (b) If ad is not the shortest side of $acbd$, the lemma conclusion might not hold.

$d \in Q_3(a)$, we have that $\angle cad > \pi/2$. This implies that $|ca| < |cd|$, which along with the fact that $a, d \in Q_3(c)$ contradict the fact that $\overrightarrow{cd} \in Y_4$. Also note that $d \notin Q_1(a)$, since in that case ab and cd could not intersect. In the following we discuss the case $d \in Q_2(a)$; the case $d \in Q_4(a)$ is symmetric.

A first observation is that c must lie below b; otherwise $|cb| < |cd|$ (since $\angle cbd > \pi/2$), which would contradict the fact that $\overrightarrow{cd} \in Y_4$. We now prove by contradiction that there is no edge in $\mathcal{P}_R(d \to a)$ crossing ab. Assume the contrary, and let $\overrightarrow{xy} \in \mathcal{P}_R(d \to a)$ be such an edge. Then necessarily $x \in R(a, d)$ and $\overrightarrow{xy} \in Q_4(x)$. Note that y cannot lie below a; otherwise $|xa| < |xy|$ (since $\angle xay > \pi/2$), which would contradict the fact that $\overrightarrow{xy} \in Y_4$. Also y must lie outside $D(c, |cd|) \cap Q(c, d)$, otherwise $\overrightarrow{cd}$ could not be in Y_4. These together show that y sits to the right of c. See Fig. 9a. Then the following inequalities regarding the quadrilateral $xayb$ must hold:

(i) $|by| > |bc|$, due to the fact that $\angle bcy > \pi/2$.

(ii) $|bx| \geq |bd|$ ($|bx| = |bd|$ if x and d coincide). If x and d are distinct, the inequality $|bx| > |bd|$ follows from the fact that $|cx| \geq |cd|$ (since x is outside $D(c, |cd|)$), and Proposition 1 applied to the quadrilateral $xcbd$:

$$|bd| + |cx| < |bx| + |cd|$$

Inequalities (i) and (ii) show that by and bx are longer than sides of the quadrilateral $acbd$, and so they must be longer than the shortest side of $acbd$, which by assumption (ii) of the lemma is ad: $\min\{|bx|, |by|\} \geq |ad| \geq |ax|$ (this latter inequality follows from the fact that $x \in R(d, a)$). Also note that $|ab| \leq |ay|$, since $\overrightarrow{ab} \in Y_4$ and y lies in the same quadrant of a as b. The fact that both diagonals of $xayb$ are in Y_4 enables us to apply Lemma 4(ii) to conclude that ay is not a shortest side of the quadrilateral $xayb$. Thus xa is a shortest side of the quadrilateral $xayb$, and we can use Lemma 4(ii) to claim that

$$|xa| < \min\{|xy|, |ab|\} \leq |xy|.$$

This contradicts our assumption that $\overrightarrow{xy} \in Y_4$.   □

Fig. 9(b) shows that the claim of the lemma might be false without assumption (ii).

6.4.   *Calculations for the stretch factor of $p(a, b)$ in Lemma 9*

We start by computing the stretch factor of the short paths claimed by statements **S2** and **S3**.

S2 If $xy \in Y_4$ and $zw \in Y_4$ are short, and if xy intersects zw, then there is a short path P between any two of the endpoints of these edges, of length

$$|P| \leq |xy| + |zw| + 3(2 + \sqrt{2}) \max\{|xy|, |zw|\}. \tag{14}$$

This upper bound can be derived as follows. Let ij be a shortest side of the quadrilateral $xzyw$. By Lemma 8, Y_4 contains a path $p(i, j)$ no longer than $6(\sqrt{2}+1)|ij|$. By Lemma 4, $|ij| \leq \max\{|xy|, |zw|\}/\sqrt{2}$. These together with the fact that $|P| \leq |xy| + |zw| + |p(i, j)|$ yield inequality (14).

S3 Here we prove a tighter version of this statement: If $p(x, y)$ and $p(z, w)$ are short paths that intersect, then there is a short path P between any two of the endpoints of these paths, of length

$$|P| \leq |p(x, y)| + |p(z, w)| + 3(2 + \sqrt{2}) \max\{|xy|, |zw|\}. \tag{15}$$

This follows immediately from **S2** and the fact that no edge of $p(x, y) \cup p(z, w)$ is longer than $\max\{|xy|, |zw|\}$ (by Lemma 8).

Case 1: $\mathcal{P}_R(b \to c)$ and ac intersect. Then by **S3** we have

$$\begin{aligned}
|p(a,b)| &\le |\mathcal{P}_R(b,c)| + |ac| + 3(2+\sqrt{2})\max\{|bc|,|ac|\} \\
&\le \sqrt{2}|bc| + |ac| + 3(2+\sqrt{2})\sqrt{2}|ac| &&\text{(by (7), (11)ii)} \\
&= 3(3+2\sqrt{2})|ac| \le 3(3+2\sqrt{2})|ab| &&\text{(by (11)i).}
\end{aligned}$$

Case 2(i): $\mathcal{P}_R(b \to c)$ and ac do not intersect; $\mathcal{P}_R(b' \to a)$ and ab do not intersect; and $\mathcal{P}_R(b' \to a)$ intersects ac. By **S3**, there is a short path $p(a,b')$ of length

$$\begin{aligned}
|p(a,b')| &\le |\mathcal{P}_R(b',a)| + |ac| + 3(2+\sqrt{2})\max\{|b'a|,|ac|\} \\
&\le |b'a|\sqrt{2} + |ac| + 3(2+\sqrt{2})\max\{|b'a|,|ac|\} &&\text{(by (7)).} \quad (16)
\end{aligned}$$

Next we establish an upper bound on $|b'a|$. By the triangle inequality,

$$|ab'| < |ac| + |cb'| \ \le \ 3|ac| \qquad \text{(by (12)).} \tag{17}$$

Substituting this inequality in (16) yields

$$|p(a,b')| \le (19 + 12\sqrt{2})|ac|. \tag{18}$$

Thus $p(a,b) = p(a,b') \oplus \mathcal{P}_R^{-1}(b \to c)$ is a path in Y_4 of length

$$\begin{aligned}
|p(a,b)| &\le |p(a,b')| + |bc|\sqrt{2} &&\text{(by (7))} \\
&\le |p(a,b')| + 2|ac| &&\text{(by (11)ii)} \\
&\le (21 + 12\sqrt{2})|ac| &&\text{(by (18))} \\
&\le (21 + 12\sqrt{2})|ab| &&\text{(by (11)i).}
\end{aligned}$$

Case 2(ii): $\mathcal{P}_R(b \to c)$ and ac do not intersect; $\mathcal{P}_R(b' \to a)$ and ab do not intersect; and $\mathcal{P}_R(b' \to a)$ does not intersect ac. Then $\mathcal{P}_R(c \to b')$ must intersect $\mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$. By **S3** there is a short path $p(c,b)$ of length

$$\begin{aligned}
|p(c,b)| &\le |\mathcal{P}_R(c \to b')| + |\mathcal{P}_R(b \to c)| + |\mathcal{P}_R(b' \to a)| + 3(2+\sqrt{2})\max\{|cb'|,|bc|,|b'a|\} \\
&\le (|cb'| + |bc| + |b'a|)\sqrt{2} + 3(2+\sqrt{2})\max\{|cb'|,|bc|,|b'a|\} &&\text{(by (7)).}
\end{aligned}$$

Inequalities (11)ii, (12) and (17) imply that $\max\{|cb'|,|bc|,|b'a|\} \le 3ac$. Substituting in the above, we get

$$\begin{aligned}
|p(c,b)| &\le (2 + \sqrt{2} + 3)\sqrt{2}|ac| + 9(2+\sqrt{2})|ac| \\
&\le (20 + 14\sqrt{2})|ac| &&\text{(by (11)i).}
\end{aligned}$$

Thus $p(a,b) = ac \oplus p(c,b)$ is a path in Y_4 from a to b of length

$$|p(a,b)| \le (21 + 14\sqrt{2})|ac| \ \le \ (21 + 14\sqrt{2})|ab| \qquad \text{(by (11)i).}$$

Case 3: $\mathcal{P}_R(b \rightarrow c)$ and ac do not intersect, and $\mathcal{P}_R(b' \rightarrow a)$ intersects ab. If $\mathcal{P}_R(b' \rightarrow a)$ intersects ab at a, then $p(a,b) = \mathcal{P}_R(b \rightarrow c) \oplus \mathcal{P}_R(b' \rightarrow a)$ is clearly short and does not exceed the spanning ratio of the lemma. Otherwise, there is an edge $\overrightarrow{de} \in \mathcal{P}_R(b' \rightarrow a)$ that crosses ab, and $\mathcal{P}_R(a \rightarrow e)$ intersects $de \oplus \mathcal{P}_R(e \rightarrow a)$ (as established in the proof of Lemma 9). If $\mathcal{P}_R(a \rightarrow e)$ intersects de, then by **S3** there is a short path $p(a,e)$ of length

$$|p(a,e)| \leq |\mathcal{P}_R(a \rightarrow e)| + |de| + 3(2 + \sqrt{2}) \max\{|ae|, |de|\} \qquad (19)$$

Otherwise, if $\mathcal{P}_R(a \rightarrow e)$ intersects $\mathcal{P}_R(e \rightarrow a)$, then by **S3** there is a short path $p(a,e)$ of length

$$|p(a,e)| \leq |\mathcal{P}_R(a \rightarrow e)| + |\mathcal{P}_R(e \rightarrow a)| + 3(2 + \sqrt{2})|ae| \qquad (20)$$

A loose upper bound on $|ae|$ can be obtained by employing Proposition 1 to the quadrilateral $aebd$: $|ae| + |bd| < |ab| + |de| < |ab| + |ab'|$. Substituting the upper bound for ab' from (17) yields

$$|ae| < |ab| + 3|ac| \leq 4|ab|. \qquad (21)$$

By Lemma 2, $|de| \leq |ab'|$ (since $de \in \mathcal{P}_R(b' \rightarrow a)$), which along with (17) implies

$$|de| \leq 3|ab|. \qquad (22)$$

Substituting inequalities (7), (21) and (22) in (19) yields

$$|p(a,e)| \leq (27 + 16\sqrt{2})|ab|.$$

Substituting inequalities (7) and (21) in (20) gives

$$|p(a,e)| \leq (24 + 20\sqrt{2})|ab|,$$

which is a looser upper bound that applies to both cases. Then

$$p(a,b) = p(a,e) \oplus \mathcal{P}_R^{-1}(b' \rightarrow a) \oplus \mathcal{P}_R^{-1}(b \rightarrow c)$$

is a path from a to b of length

$$\begin{aligned}|p(a,b)| &\leq (24 + 20\sqrt{2})|ab| + 3\sqrt{2}|ab| + 2|ab| \qquad \text{(by (23), (17), (11))} \\ &= (26 + 23\sqrt{2})|ab|.\end{aligned}$$